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**THEORETICAL STUDIES OF A MOLECULAR
BEAM GENERATOR**

By

John H. Heinbockel, Principal Investigator

Progress Report

For the period May 16, 1992 to November 15, 1992

Prepared for

National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23681-2199

Under

Research Grant NAG-1-1424

Dr. Sang H. Choi, Technical Monitor
SSD-High Energy Science Branch

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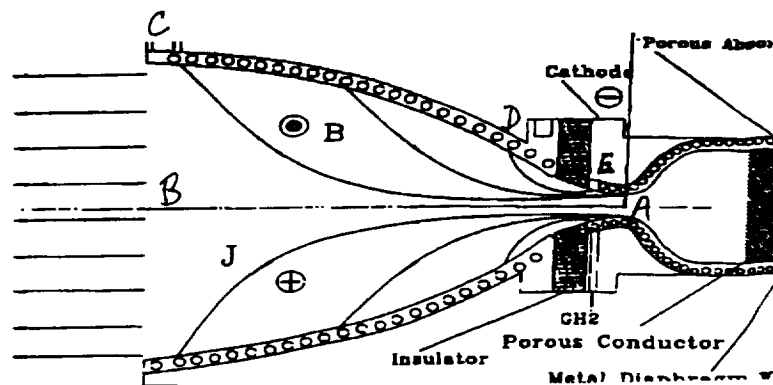
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THEORETICAL STUDIES OF A MOLECULAR BEAM GENERATOR

The following is a proposed baseline model that is being developed for the simulation of hydrodynamic generator, which can be converted at a later date to a magnetohydrodynamic MHD thruster by adding the necessary electric and magnetic fields. The following development will include the electric and magnetic terms, however, the initial computer program will not include these terms. The analysis that follows is for a one species, single temperature model constructed over the domain D defined by the region enclosed by ABCDEF illustrated in the figure 1.



CONTINUITY EQUATION

The continuity equation expresses conservation of mass and is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \quad (1)$$

where $\rho = \rho(r, \theta, z, t)$ is the density of the gas, and $\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z$ is the velocity. In cylindrical coordinates the equation (1) has the form

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(r\rho V_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho V_\theta)}{\partial \theta} + \frac{\partial(\rho V_z)}{\partial z} = 0. \quad (2)$$

CONSERVATION OF MOMENTUM

The equation for conservation of linear momentum is given by

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho(\vec{V} \cdot \nabla) \vec{V} = \sum_{i=1}^n \vec{F}_i \quad (3)$$

where $\sum_{i=1}^n \vec{F}_i$ represents a summation of body forces per unit volume acting upon a control volume within the domain D. We consider initially the pressure force

$$\vec{F}_1 = -\nabla P \quad (4)$$

where pressure and density are related by the equation of state gas law $P = \rho^* RT$, where ρ^* is the density in mole/m³. i.e. $\rho^* W = \rho$ where W is the molecular weight in kg/mole. The force due to viscosity is

$$\vec{F}_2 = \eta \left\{ \nabla^2 \vec{V} + \nabla(\nabla \cdot \vec{V}) \right\} - \frac{2}{3} \nabla(\eta \nabla \cdot \vec{V}) + 2(\nabla \eta \cdot \nabla) \vec{V} + \nabla \eta \times (\nabla \times \vec{V}) \quad (5)$$

where

$$\eta = 1.25 \cdot 10^{-19} \frac{5}{8n_i A} \left(\frac{M m k_B T}{\pi} \right)^{1/2} \left(\frac{2k_B T}{e^2} \right)^2 = C_1 T^{5/2} \quad (6)$$

is the plasma viscosity, with

$$A = 2(\log(1 + \alpha^2) - \frac{\alpha^2}{1 + \alpha^2})$$

a constant which depends upon the ionization factor α , ($\alpha = 1$ for a fully ionized gas). The additional constants are M the ion mass (Kg), m the electron mass, e the electron charge, k_B Boltzmann's constant, T is the absolute temperature, and $n_i = 1$ for a singly ionized plasma. For $\alpha = 0$, we employ an empirical curve fit for the viscosity as a function of temperature. The magnetic force is given by

$$\vec{F}_3 = \vec{J} \times \vec{B} \quad (7)$$

where \vec{J} is the current density. The gravitational force is given by

$$\vec{F}_4 = \rho \vec{g}.$$

The electric force is given by

$$\vec{F}_5 = e \vec{E}.$$

All additional forces are represented by

$$\sum_{i=6}^n \vec{F}_i$$

and are neglected for the present.

CONSERVATION OF ENERGY

Representing the internal energy by $u = C_p T$ where C_p is the specific heat at constant pressure and T is the absolute temperature, the energy equation can be written in the form of an energy balance as

$$\rho \frac{\partial(C_p T)}{\partial t} + \rho(\vec{V} \cdot \nabla)(C_p T) = \nabla(K_T \nabla T) + \sum_{i=1}^n \phi_i \quad (8)$$

where both the specific heat C_p and thermal conductivity K_T are treated as functions of temperature T . The thermal conductivity K_T of the medium is given by the Spitzer-Harris relation

$$K_T = \frac{4.4 \cdot 10^{-10} T^{5/2}}{23 - \log \left[\frac{1.22 \cdot 10^3 n^{1/2}}{T^{3/2}} \right]} \quad (9)$$

and n is the plasma number density in particles/ m^3 . In addition to the heat loss term the right hand side of equation (8) contains the terms

$$\phi_1 = (\vec{E} + \vec{V} \times \vec{B}) \cdot \vec{J} - \eta \vec{V} \cdot \vec{E} \quad (10)$$

which represents joule heating,

$$\begin{aligned} \phi_2 = \eta \left\{ 2 \left[\left(\frac{\partial V_r}{\partial r} \right)^2 + \left(\frac{V_r}{r} \right)^2 + \left(\frac{\partial V_z}{\partial z} \right)^2 \right] \right. \\ \left. + \left(\frac{\partial V_\theta}{\partial z} \right)^2 + \left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r} \right)^2 + \left(\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r} \right)^2 \right\} \\ + \lambda \left(\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z} \right)^2 \end{aligned} \quad (11)$$

which represents viscous dissipation. Here η and λ are viscosity coefficients satisfying $\lambda + \frac{2}{3}\eta = 0$. In addition there is the radiation loss term. Various forms of the radiation term exists in the literature. As a first approximation we take the radiation loss term from reference 11 which can be expressed

$$\phi_3 = \nabla \lambda_R \nabla \left(\frac{4}{3} \sigma T^4 \right)$$

where λ_R is the Rosseland mean free path ($\lambda_R = 1/\alpha_R$ where α_R is the Rosseland absorption coefficient (cm^{-1})), and σ is the Stephan Boltzman constant. The remaining terms $\sum_{i=4}^n \phi_i$ represents additional energy considerations which are initially neglected.

MAXWELL'S EQUATIONS

Maxwell's equations in the MKS Rational system of units can be expressed

$$\text{Gauss's law for magnetism} \quad \nabla \cdot \vec{B} = 0 \quad (12)$$

$$\text{Gauss's law for electricity (Coulomb's law)} \quad \nabla \cdot \vec{D} = \rho_e \quad (13)$$

$$\text{Ampere's law} \quad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (14)$$

$$\text{Faraday's law} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (15)$$

CONSTITUTIVE EQUATIONS

Assuming an isotropic, homogeneous medium we adopt the constitutive equations

$$\vec{D} = \epsilon \vec{E} \quad \text{and} \quad \vec{B} = \mu \vec{H}.$$

OHM'S LAW

Ohm's law is written in the form

$$\vec{J} = \sigma (\vec{E} + \vec{V} \times \vec{B}) - \frac{\Omega}{B} (\vec{J} \times \vec{B}) \quad (16)$$

where \vec{J} is the current density, \vec{E} is the electric field, \vec{B} is the induced magnetic field and σ is the electric conductivity with units of mho/m and Ω is the Hall parameter given by (reference 1)

$$\Omega = 9.6(10^{16})(T^{3/2}B/Zn \log \Lambda)$$

with Coulomb logarithm given by

$$\log \Lambda \approx 23 - \log(1.22 \times 10^3 n^{1/2} / T^{3/2})$$

with n the plasma number density.

ELECTROMAGNETIC FIELD EQUATIONS

Neglecting the displacement current modifies Ampere's law to

$$\nabla \times \vec{B} = \mu \vec{J}. \quad (17)$$

Assuming that the charge density is constant implies the equation of continuity of charge is $\nabla \cdot \vec{J} = 0$. (Note that the divergence of equation (17) also gives this result.) From Jackson, reference 10, along with neglecting the displacement current, it is appropriate to ignore Coulomb's law as its effects are negligible. We thus obtain the electromagnetic field equations

$$\begin{aligned} \nabla \times \vec{B} &= \mu \vec{J} \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned} \quad (18)$$

Using equation (17) in Ohm's law we solve for \vec{E} and write

$$\vec{E} = \alpha \nabla \times \vec{B} - \vec{V} \times \vec{B} + \beta (\nabla \times \vec{B}) \times \vec{B} \quad (19)$$

where $\beta = \Omega / \mu B \sigma$ and $\alpha = 1 / \sigma \mu$. We substitute the results from equation (19) into Faraday's law and write

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times (\alpha \nabla \times \vec{B}) - \nabla \times (\vec{V} \times \vec{B}) + \nabla \times \beta (\nabla \times \vec{B}) \times \vec{B} \quad (20)$$

and since β is a function of T we find

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times (\alpha \nabla \times \vec{B}) - \nabla \times (\vec{V} \times \vec{B}) + \nabla \beta \times (\nabla \times \vec{B}) \times \vec{B} + \beta \nabla \times (\nabla \times \vec{B}) \times \vec{B} \quad (21)$$

SUMMARY OF BASIC EQUATIONS USED FOR MODELING

Continuity

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{V}) = 0$$

Momentum

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho(\vec{V} \cdot \nabla) \vec{V} = \sum_{i=1}^n \vec{F}_i$$

Energy

$$\rho \frac{\partial (C_p T)}{\partial t} + \rho(\vec{V} \cdot \nabla)(C_p T) = \nabla(K_T \nabla T) + \sum_{i=1}^n \phi_i$$

Electromagnetic field equations

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times (\alpha \nabla \times \vec{B}) - \nabla \times (\vec{V} \times \vec{B}) + \nabla \times \beta(\nabla \times \vec{B}) \times \vec{B}$$

This produces a system of eight simultaneous partial differential equations in the eight unknowns

$$B_r, B_\theta, B_z, \rho, V_r, V_\theta, V_z, T.$$

Throughout the calculations the following quantities can be generated in terms of the above variables.

$$\begin{aligned} \vec{J} &= \frac{1}{\mu} \nabla \times \vec{B} \\ \vec{E} &= \alpha \mu \vec{J} - \vec{V} \times \vec{B} + \mu \beta(\vec{J} \times \vec{B}) \end{aligned} \tag{22}$$

SCALAR FORM OF FIELD EQUATIONS

Assuming symmetry with respect to the θ variable, all derivatives with respect to θ are neglected. The following set of scalar equations then results

Continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial (r \rho V_r)}{\partial r} + \frac{\partial (\rho V_z)}{\partial z} = 0$$

Momentum

$$\begin{aligned}\rho \frac{\partial V_r}{\partial t} + \rho \left(V_r \frac{\partial V_r}{\partial r} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} \right) &= \sum_{i=1}^n (\vec{F}_i)_r \\ \rho \frac{\partial V_\theta}{\partial t} + \rho \left(V_r \frac{\partial V_\theta}{\partial r} + V_z \frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r} \right) &= \sum_{i=1}^n (\vec{F}_i)_\theta \\ \rho \frac{\partial V_z}{\partial t} + \rho \left(V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} \right) &= \sum_{i=1}^n (\vec{F}_i)_z\end{aligned}$$

Energy

$$\begin{aligned}\rho \left(C_p + T \frac{\partial C_p}{\partial T} \right) \frac{\partial T}{\partial t} + \rho V_r \left(C_p + T \frac{\partial C_p}{\partial T} \right) \frac{\partial T}{\partial r} + \rho V_z \left(C_p + T \frac{\partial C_p}{\partial T} \right) \frac{\partial T}{\partial z} = \\ \frac{\partial K_T}{\partial T} \left(\left(\frac{\partial T}{\partial r} \right)^2 + \left(\frac{\partial T}{\partial z} \right)^2 \right) + K_T \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right] + \sum_{i=1}^n \phi_i\end{aligned}$$

Electromagnetic field equations

$$\begin{aligned}
-\frac{\partial B_r}{\partial t} &= -\alpha \frac{\partial^2 B_r}{\partial z^2} + \alpha \frac{\partial^2 B_z}{\partial r \partial z} + V_z \frac{\partial B_r}{\partial z} + B_r \frac{\partial V_z}{\partial z} - V_r \frac{\partial B_z}{\partial z} - B_z \frac{\partial V_r}{\partial z} \\
&\quad - \beta \left[B_z \frac{\partial^2 B_\theta}{\partial z^2} + \frac{\partial B_z}{\partial z} \frac{\partial B_\theta}{\partial z} + B_r \left(\frac{\partial^2 B_\theta}{\partial r \partial z} + \frac{1}{r} \frac{\partial B_\theta}{\partial z} \right) + \frac{\partial B_r}{\partial z} \left(\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \right) \right] \\
&\quad + \frac{\partial \beta}{\partial z} \left[B_z \frac{\partial B_\theta}{\partial z} + B_r \left(\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \right) \right] \\
-\frac{\partial B_\theta}{\partial t} &= -\alpha \frac{\partial^2 B_\theta}{\partial z^2} - \alpha \frac{\partial^2 B_\theta}{\partial r^2} - \frac{\alpha}{r} \frac{\partial B_\theta}{\partial r} + \frac{\alpha}{r^2} B_\theta - V_\theta \frac{\partial B_z}{\partial z} - B_z \frac{\partial V_\theta}{\partial z} \\
&\quad + V_z \frac{\partial B_\theta}{\partial z} + B_\theta \frac{\partial V_z}{\partial z} + V_r \frac{\partial B_\theta}{\partial r} + B_\theta \frac{\partial V_r}{\partial r} - V_\theta \frac{\partial B_r}{\partial r} - B_r \frac{\partial V_\theta}{\partial r} \\
&\quad + \beta \left[B_z \left(\frac{\partial^2 B_r}{\partial z^2} - \frac{\partial^2 B_z}{\partial r \partial z} \right) + \frac{\partial B_z}{\partial z} \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) - \frac{2B_\theta}{r} \frac{\partial B_\theta}{\partial z} \right. \\
&\quad \left. + B_r \left(\frac{\partial^2 B_r}{\partial r \partial z} - \frac{\partial^2 B_z}{\partial r^2} \right) + \frac{\partial B_r}{\partial r} \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) \right] + \frac{\partial \beta}{\partial r} \left[B_\theta \frac{\partial B_\theta}{\partial z} + B_r \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) \right] \\
&\quad + \frac{\partial \beta}{\partial z} \left[B_z \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) - B_\theta \left(\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \right) \right] \\
-\frac{\partial B_z}{\partial t} &= \alpha \frac{\partial^2 B_r}{\partial r \partial z} - \alpha \frac{\partial^2 B_z}{\partial r^2} + \frac{\alpha}{r} \frac{\partial B_r}{\partial z} - \frac{\alpha}{r} \frac{\partial B_z}{\partial r} - V_z \frac{\partial B_r}{\partial r} - B_r \frac{\partial V_z}{\partial r} \\
&\quad + V_r \frac{\partial B_z}{\partial r} + B_z \frac{\partial V_r}{\partial r} + \frac{1}{r} (V_r B_z - V_z B_r) \\
&\quad + \beta \left[B_z \frac{\partial^2 B_\theta}{\partial r \partial z} + \frac{\partial B_z}{\partial r} \frac{\partial B_\theta}{\partial z} + B_r \left(\frac{\partial^2 B_\theta}{\partial r^2} + \frac{2}{r} \frac{\partial B_\theta}{\partial r} \right) + \frac{\partial B_r}{\partial r} \left(\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \right) + \frac{B_z}{r} \frac{\partial B_\theta}{\partial z} \right] \\
&\quad + \frac{\partial \beta}{\partial r} \left[B_z \frac{\partial B_\theta}{\partial z} + B_r \left(\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \right) \right]
\end{aligned}$$

These equations are subject to certain boundary and initial conditions which are now discussed.

BOUNDARY AND INITIAL CONDITIONS

With reference to the figure 1, the line \overline{AF} has the input conditions

$$\rho = \rho_0 = \text{constant}$$

$$T = T_0 = \text{constant}$$

$$V_r = V_\theta = 0$$

$$V_z = V_0 = \text{constant}$$

$$\frac{\partial B_r}{\partial z} = \frac{\partial B_\theta}{\partial z} = \frac{\partial B_z}{\partial z}$$

Due to symmetry considerations the line \overline{AB} has the center line boundary conditions

$$\begin{aligned}\frac{\partial \rho}{\partial r} &= 0 \\ \frac{\partial T}{\partial r} &= 0 & \frac{\partial B_r}{\partial r} &= 0 \\ \frac{\partial V_r}{\partial r} &= 0 & \frac{\partial B_\theta}{\partial r} &= 0 \\ \frac{\partial V_\theta}{\partial r} &= 0 & \frac{\partial B_z}{\partial r} &= 0 \\ \frac{\partial V_z}{\partial r} &= 0\end{aligned}$$

The far field conditions along the line \overline{BC} are given by

$$\begin{aligned}\frac{\partial \rho}{\partial z} &= 0 & \frac{\partial V_z}{\partial z} &= 0 \\ \frac{\partial T}{\partial z} &= 0 & \frac{\partial B_r}{\partial z} &= 0 \\ \frac{\partial V_r}{\partial z} &= 0 & \frac{\partial B_\theta}{\partial z} &= 0 \\ \frac{\partial V_\theta}{\partial z} &= 0 & \frac{\partial B_z}{\partial z} &= 0\end{aligned}$$

For the insulated boundary segment between \overline{ED} we assign the boundary conditions

$$\begin{aligned}V_r = V_\theta = V_z &= 0 & \text{no slip boundary condition} \\ T &= T_0 = \text{constant} \\ \frac{\partial \rho}{\partial r} &= 0 \\ B_r = B_\theta = B_z &= 0\end{aligned}$$

For the uninsulated boundary segments \overline{FE} and \overline{DC} we assign the conditions

$$\begin{aligned}V_r = V_\theta = V_z &= 0 & \text{no slip boundary condition} \\ T &= T_0 = \text{constant} \\ \frac{\partial \rho}{\partial r} &= 0 \\ J_\theta = 0 \text{ which implies } \frac{\partial B_r}{\partial z} &= \frac{\partial B_z}{\partial r} & \text{and simultaneously} \\ \frac{\partial(rB_\theta)}{\partial r} &= 0 & \text{and } \frac{\partial B_\theta}{\partial z} = 0\end{aligned}$$

These later boundary conditions upon \vec{B} insures that the electric field \vec{E} satisfies the condition $\vec{E} \cdot \vec{t} = 0$ everywhere on the nozzle boundary, where \vec{t} represents a unit tangent vector to an arbitrary point on the nozzle boundary.

Initial conditions are assigned in order to avoid large initial transients in the numerical solution because large changes can lead to numerical instabilities of the system of partial differential equations. We therefore assign the following initial conditions at all interior grid points.

$$T = T_0 = \text{constant}$$

$$V_r = V_\theta = 0$$

$$V_z = V_0 = \text{constant}$$

$$B_r, B_\theta \text{ and } B_z \text{ are assigned values such that } \nabla \cdot \vec{B} = 0$$

everywhere in the solution domain.

TRANSFORMATION OF COORDINATES

We make the change of variables

$$x = \frac{z}{b} \quad y = \frac{r}{f(z)}$$

so that the domain of the nozzle $0 \leq r \leq f(z)$, $0 \leq z \leq b$ transforms to the computation domain $0 \leq x \leq 1$ and $0 \leq y \leq 1$. The scalar form of the governing equations can then be written in the following forms.

Continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{f(z)} \frac{\partial(\rho V_r)}{\partial y} + \frac{\rho V_r}{y f(z)} + \frac{1}{b} \frac{\partial(\rho V_z)}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial(\rho V_z)}{\partial y} = 0$$

Momentum Equations

$$\frac{\partial V_r}{\partial t} + \frac{V_r}{f(z)} \frac{\partial V_r}{\partial y} + V_z \left(\frac{1}{b} \frac{\partial V_r}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial V_r}{\partial y} \right) - \frac{V_\theta^2}{y f(z)} = \frac{1}{\rho} \sum_{i=1}^n (\vec{F}_i)_r$$

$$\frac{\partial V_\theta}{\partial t} + \frac{V_r}{f(z)} \frac{\partial V_\theta}{\partial y} + V_z \left(\frac{1}{b} \frac{\partial V_\theta}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial V_\theta}{\partial y} \right) + \frac{V_r V_\theta}{y f(z)} = \frac{1}{\rho} \sum_{i=1}^n (\vec{F}_i)_\theta$$

$$\frac{\partial V_z}{\partial t} + \frac{V_r}{f(z)} \frac{\partial V_z}{\partial y} + V_z \left(\frac{1}{b} \frac{\partial V_z}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial V_z}{\partial y} \right) = \frac{1}{\rho} \sum_{i=1}^n (\vec{F}_i)_z$$

Energy Equation

$$\begin{aligned} \rho(C_p + T \frac{\partial C_p}{\partial T}) \left(\frac{\partial T}{\partial t} + \frac{V_r}{f(z)} \frac{\partial T}{\partial y} + V_z \left(\frac{1}{b} \frac{\partial T}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial T}{\partial y} \right) \right) = \\ K_T \left(\frac{1}{b^2} \frac{\partial^2 T}{\partial x^2} - \frac{2y f'(z)}{b f(z)} \frac{\partial^2 T}{\partial x \partial y} - \left[y \left(\frac{f''(z)}{f(z)} - 2 \left(\frac{f'(z)}{f(z)} \right)^2 \right) \right] \frac{\partial T}{\partial y} \right. \\ \left. + y^2 \left(\frac{f'(z)}{f(z)} \right)^2 \frac{\partial^2 T}{\partial y^2} + \frac{1}{y f^2(z)} \frac{\partial T}{\partial y} + \frac{1}{f^2(z)} \frac{\partial^2 T}{\partial y^2} \right) + \sum_{i=1}^n \phi_i \end{aligned}$$

Electromagnetic Field Equations

\hat{e}_r -component

$$\begin{aligned} - \frac{\partial B_r}{\partial t} = -\alpha \left\{ \frac{1}{b^2} \frac{\partial^2 b_r}{\partial x^2} - \frac{2y f'(z)}{b f(z)} \frac{\partial^2 B_r}{\partial x \partial y} + \left(\frac{y f'(z)}{f(z)} \right)^2 \frac{\partial^2 B_r}{\partial y^2} \right. \\ \left. - y \left(\frac{f''(z)}{f(z)} - 2 \left(\frac{f'(z)}{f(z)} \right)^2 \right) \frac{\partial B_r}{\partial y} \right\} \\ + \alpha \left\{ - \frac{f'(z)}{f^2(z)} \frac{\partial B_z}{\partial y} + \frac{1}{b f(z)} \frac{\partial^2 B_z}{\partial x \partial y} - \frac{y f'(z)}{f^2(z)} \frac{\partial^2 B_z}{\partial y^2} \right\} \\ + V_z \left(\frac{1}{b} \frac{\partial B_r}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial B_r}{\partial y} \right) + B_r \left(\frac{1}{b} \frac{\partial V_z}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial V_z}{\partial y} \right) \\ - V_r \left(\frac{1}{b} \frac{\partial B_z}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial B_z}{\partial y} \right) - B_z \left(\frac{1}{b} \frac{\partial V_r}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial V_r}{\partial y} \right) \\ - \beta \left\{ B_z \left(\frac{1}{b^2} \frac{\partial^2 B_\theta}{\partial x^2} - \frac{2y f'(z)}{b f(z)} \frac{\partial^2 B_\theta}{\partial x \partial y} + \left(\frac{y f'(z)}{f(z)} \right)^2 \frac{\partial^2 B_\theta}{\partial y^2} - y \left[\frac{f''(z)}{f(z)} - 2 \left(\frac{f'(z)}{f(z)} \right)^2 \right] \frac{\partial B_\theta}{\partial y} \right) \right. \\ + \left(\frac{1}{b} \frac{\partial B_z}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial B_z}{\partial y} \right) \left(\frac{1}{b} \frac{\partial B_\theta}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial B_\theta}{\partial y} \right) \\ + B_r \left(- \frac{f'(z)}{f^2(z)} \frac{\partial B_\theta}{\partial y} + \frac{1}{b f(z)} \frac{\partial^2 B_\theta}{\partial x \partial y} - \frac{y f'(z)}{f(z)} \frac{\partial^2 B_\theta}{\partial y^2} + \frac{1}{y f(z)} \left(\frac{1}{b} \frac{\partial B_\theta}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial B_\theta}{\partial y} \right) \right) \\ + \left(\frac{1}{b} \frac{\partial B_r}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial B_r}{\partial y} \right) \left(\frac{1}{f(z)} \frac{\partial B_\theta}{\partial y} + \frac{B_\theta}{y f(z)} \right) \left. \right\} \\ - \beta'(T) \left(\frac{1}{b} \frac{\partial T}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial T}{\partial y} \right) \left(B_z \left(\frac{1}{b} \frac{\partial B_\theta}{\partial x} - \frac{y f'(z)}{f(z)} \frac{\partial B_\theta}{\partial y} \right) + B_r \left(\frac{1}{f(z)} \frac{\partial B_\theta}{\partial y} + \frac{B_\theta}{y f(z)} \right) \right) \end{aligned}$$

\hat{e}_θ -component

$$\begin{aligned}
-\frac{\partial B_\theta}{\partial t} = & -\alpha \left(\frac{1}{b^2} \frac{\partial^2 B_\theta}{\partial x^2} - \frac{2yf'(z)}{bf(z)} \frac{\partial^2 B_\theta}{\partial x \partial y} + \left(\frac{yf'(z)}{f(z)} \right)^2 \frac{\partial^2 B_\theta}{\partial y^2} \right. \\
& - y \left[\frac{f''(z)}{f(z)} - 2 \left(\frac{f'(z)}{f(z)} \right)^2 \right] \frac{\partial B_\theta}{\partial y} - \frac{1}{f^2(z)} \frac{\partial^2 B_\theta}{\partial y^2} - \frac{1}{yf^2(z)} \frac{\partial B_\theta}{\partial y} + \frac{B_\theta}{y^2 f^2(z)} \Bigg) \\
& - V_\theta \left(\frac{1}{b} \frac{\partial B_z}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_z}{\partial y} \right) - B_z \left(\frac{1}{b} \frac{\partial V_\theta}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial V_\theta}{\partial y} \right) + V_z \left(\frac{1}{b} \frac{\partial B_\theta}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_\theta}{\partial y} \right) \\
& + B_\theta \left(\frac{1}{b} \frac{\partial V_z}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial V_z}{\partial y} \right) + \frac{V_r}{f(z)} \frac{\partial B_\theta}{\partial y} + \frac{B_\theta}{f(z)} \frac{\partial V_r}{\partial y} - \frac{V_\theta}{f(z)} \frac{\partial B_r}{\partial y} - \frac{B_z}{f(z)} \frac{\partial V_\theta}{\partial y} \\
& \beta \left\{ B_z \left[\frac{1}{b^2} \frac{\partial^2 B_r}{\partial x^2} - \frac{2yf'(z)}{bf(z)} \frac{\partial^2 B_r}{\partial x \partial y} + \left(\frac{yf'(z)}{f(z)} \right)^2 \frac{\partial^2 B_r}{\partial y^2} - y \left[\frac{f''(z)}{f(z)} - 2 \left(\frac{f'(z)}{f(z)} \right)^2 \right] \frac{\partial B_r}{\partial y} \right. \right. \\
& \left. \left. + \frac{f'(z)}{f^2(z)} \frac{\partial B_z}{\partial y} - \frac{1}{bf(z)} \frac{\partial^2 B_z}{\partial x \partial y} - \frac{yf'(z)}{f^2(z)} \frac{\partial^2 B_z}{\partial y^2} \right] \right. \\
& \left(\frac{1}{b} \frac{\partial B_z}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_z}{\partial y} \right) \left(\frac{1}{b} \frac{\partial B_r}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_r}{\partial y} - \frac{1}{f(z)} \frac{\partial B_z}{\partial y} \right) \\
& - \frac{2B_\theta}{yf(z)} \left(\frac{1}{b} \frac{\partial B_\theta}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_\theta}{\partial y} \right) \\
& + B_r \left(-\frac{f'(z)}{f^2(z)} \frac{\partial B_r}{\partial y} + \frac{1}{bf(z)} \frac{\partial^2 B_r}{\partial x \partial y} - \frac{yf'(z)}{f^2(z)} \frac{\partial^2 B_r}{\partial y^2} - \frac{1}{f^2(z)} \frac{\partial^2 B_z}{\partial y^2} \right) \\
& + \left(\frac{1}{f(z)} \frac{\partial B_r}{\partial y} \right) \left(\frac{1}{b} \frac{\partial B_r}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_r}{\partial y} - \frac{1}{f(z)} \frac{\partial B_z}{\partial y} \right) \Bigg\} \\
& - \frac{\beta'(T)}{f(z)} \frac{\partial T}{\partial y} \left(-B_\theta \left[\frac{1}{b} \frac{\partial B_\theta}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_\theta}{\partial y} \right] - B_r \left[\frac{1}{b} \frac{\partial B_r}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_r}{\partial y} - \frac{1}{f(z)} \frac{\partial B_z}{\partial y} \right] \right) \\
& + \beta'(T) \left(\frac{1}{b} \frac{\partial T}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial T}{\partial y} \right) \left[B_z \left(\frac{1}{b} \frac{\partial B_r}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_r}{\partial y} - \frac{1}{f(z)} \frac{\partial B_r}{\partial y} \right) \right. \\
& \left. - B_\theta \left(\frac{1}{f(z)} \frac{\partial B_\theta}{\partial y} + \frac{B_\theta}{yf(z)} \right) \right]
\end{aligned}$$

e_z -component

$$\begin{aligned}
-\frac{\partial B_z}{\partial t} = & \alpha \left(-\frac{f'(z)}{f^2(z)} \frac{\partial B_r}{\partial y} + \frac{1}{bf(z)} \frac{\partial^2 B_r}{\partial x \partial y} - \frac{yf'(z)}{f^2(z)} \frac{\partial^2 B_r}{\partial y^2} \right. \\
& - \frac{1}{f^2(z)} \frac{\partial^2 B_z}{\partial y^2} + \frac{1}{yf(z)} \left[\frac{1}{b} \frac{\partial B_r}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_r}{\partial y} \right] - \frac{1}{yf^2(z)} \frac{\partial B_z}{\partial y} \Big) \\
& - \frac{V_z}{f(z)} \frac{\partial B_r}{\partial y} - \frac{B_r}{f(z)} \frac{\partial V_z}{\partial y} + \frac{V_r}{f(z)} \frac{\partial B_z}{\partial y} + \frac{B_z}{f(z)} \frac{\partial V_r}{\partial y} - \frac{V_z B_r}{yf(z)} + \frac{V_r B_z}{yf(z)} \\
& + \beta \left\{ B_z \left(-\frac{f'(z)}{f^2(z)} \frac{\partial B_\theta}{\partial y} + \frac{1}{bf(z)} \frac{\partial^2 B_\theta}{\partial x \partial y} - \frac{yf'(z)}{f(z)} \frac{\partial^2 B_\theta}{\partial y^2} \right) \right. \\
& + \frac{1}{f(z)} \frac{\partial B_z}{\partial y} \left(\frac{1}{b} \frac{\partial B_\theta}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_\theta}{\partial y} \right) \\
& + \frac{B_r}{f^2(z)} \frac{\partial^2 B_\theta}{\partial y^2} + \frac{B_r}{yf^2(z)} \frac{\partial B_\theta}{\partial y} + \frac{1}{f(z)} \frac{\partial B_r}{\partial y} \left(\frac{1}{f(z)} \frac{\partial B_\theta}{\partial y} + \frac{B_\theta}{yf(z)} \right) \\
& + \frac{B_z}{yf(z)} \left(\frac{1}{b} \frac{\partial B_\theta}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_\theta}{\partial y} \right) + \frac{B_r}{yf^2(z)} \frac{\partial B_\theta}{\partial y} \Big\} \\
& + \frac{\beta'(T)}{f(z)} \frac{\partial T}{\partial y} \left[B_z \left(\frac{1}{b} \frac{\partial B_\theta}{\partial x} - \frac{yf'(z)}{f(z)} \frac{\partial B_\theta}{\partial y} \right) + B_r \left(\frac{1}{f(z)} \frac{\partial B_\theta}{\partial y} + \frac{B_\theta}{yf(z)} \right) \right]
\end{aligned}$$

NUMERICAL SOLUTION

We are primarily interested in the steady state solutions and the time necessary to achieve steady state. The above system of coupled nonlinear partial differential equations are simplified by assuming symmetry with respect to the θ variable. This enables us to set all derivatives with respect to θ equal to zero. Additional assumptions regarding magnitudes of force terms and energy terms can be made by doing a dimensional analysis of the resulting system of equations. A grid generation technique is used to alter the solution domain to a rectangle. Then the equations and rectangular boundary can be scaled before any numerical solution techniques are applied. The system of equations are then solved numerically using ADI (Alternating Direct Implicit) techniques patterned after the Lax modification.

GRID GENERATION

Let $r = f(z)$ denote the nozzle boundary for $0 \leq z \leq b$ and consider the mapping from the (r, z) real coordinates to the (x, y) computational coordinates given by the transformation equations

$$\begin{aligned}
x &= \frac{z}{b} \\
y &= \frac{r}{r_{max}} = \frac{r}{f(z)}
\end{aligned} \tag{23}$$

where $r_{max} = f(z)$ denotes the nozzle boundary which changes with position. This mapping converts the region

$$D = \{r, z \mid 0 \leq z \leq b, \quad 0 \leq r \leq r_{max}\}$$

to the region D' of computational coordinates given by

$$D' = \{x, y \mid 0 \leq x \leq 1, \quad 0 \leq y \leq 1\}.$$

To handle large gradients in any of the independent variables, the computational x, y domain is partitioned into 6 regions as illustrated in the figure 2.

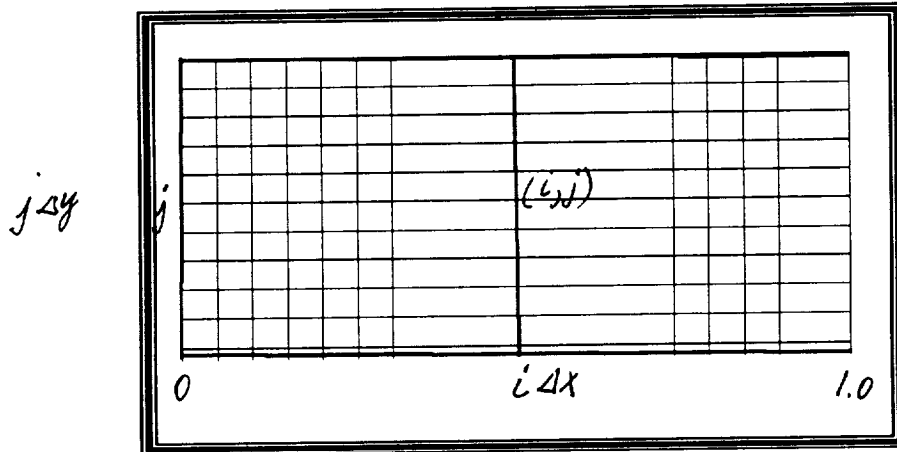


Figure 2. Computational coordinates

The 6 regions are characterized by the selection of the Δx and Δy step sizes. In this way finer grids can be specified near the boundaries and nozzle regions where large gradients can occur.

Observe that any partial differential equation of the form

$$\frac{\partial u}{\partial t} = D_1(r, z) \frac{\partial^2 u}{\partial r^2} + D_2(r, z) \frac{\partial^2 u}{\partial r \partial z} + D_3(r, z) \frac{\partial^2 u}{\partial z^2} + D_4(r, z) \frac{\partial u}{\partial r} + D_5(r, z) \frac{\partial u}{\partial z} + D_6 \quad (24)$$

where $u = u(r, z, t)$ and $D_6 = D_6(r, z, t, u, \dots)$ can be converted to computational coordinates x, y by using the chain rule for partial derivatives. These changes can be represented

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} \\ \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial z}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} &= \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial r^2} + \frac{\partial x}{\partial r} \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial r} \right] \\ &\quad + \frac{\partial u}{\partial y} \frac{\partial^2 y}{\partial r^2} + \frac{\partial y}{\partial r} \left[\frac{\partial^2 u}{\partial y \partial x} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial r} \right] \\ \frac{\partial^2 u}{\partial z^2} &= \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial z^2} + \frac{\partial x}{\partial z} \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial z} \right] \\ &\quad + \frac{\partial u}{\partial y} \frac{\partial^2 y}{\partial z^2} + \frac{\partial y}{\partial z} \left[\frac{\partial^2 u}{\partial y \partial x} \frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial z} \right] \\ \frac{\partial^2 u}{\partial r \partial z} &= \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial r \partial z} + \frac{\partial x}{\partial r} \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial z} \right] \\ &\quad + \frac{\partial u}{\partial y} \frac{\partial^2 y}{\partial r \partial z} + \frac{\partial y}{\partial r} \left[\frac{\partial^2 u}{\partial y \partial x} \frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial z} \right]\end{aligned}$$

Then the partial differential equation (24) can then be written in the x, y computational coordinates

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_1 \left[\frac{\partial u}{\partial x} x_{rr} + \frac{\partial^2 u}{\partial x^2} (x_r)^2 + \frac{\partial^2 u}{\partial x \partial y} x_r y_r + \frac{\partial u}{\partial y} y_{rr} + \frac{\partial^2 u}{\partial x \partial y} y_r x_r + \frac{\partial^2 u}{\partial y^2} (y_r)^2 \right] \\ &\quad + D_2 \left[\frac{\partial u}{\partial x} x_{rz} + \frac{\partial^2 u}{\partial x^2} x_r x_z + \frac{\partial^2 u}{\partial x \partial y} x_r y_z + \frac{\partial u}{\partial y} y_{rz} + \frac{\partial^2 u}{\partial x \partial y} y_r x_z + \frac{\partial u}{\partial y} y_r y_z \right] \\ &\quad + D_3 \left[\frac{\partial u}{\partial x} x_{zz} + \frac{\partial^2 u}{\partial x^2} (x_z)^2 + \frac{\partial^2 u}{\partial x \partial y} x_z y_z + \frac{\partial u}{\partial y} y_{zz} + \frac{\partial^2 u}{\partial x \partial y} y_z x_z + \frac{\partial^2 u}{\partial y^2} (y_z)^2 \right] \\ &\quad + D_4 \left[\frac{\partial u}{\partial x} x_r + \frac{\partial u}{\partial y} y_r \right] + D_5 \left[\frac{\partial u}{\partial x} x_z + \frac{\partial u}{\partial y} y_z \right] + D_6\end{aligned}$$

or

$$\frac{\partial u}{\partial t} = D_1^* \frac{\partial^2 u}{\partial x^2} + D_2^* \frac{\partial^2 u}{\partial x \partial y} + D_3^* \frac{\partial^2 u}{\partial y^2} + D_4^* \frac{\partial u}{\partial x} + D_5^* \frac{\partial u}{\partial y} + D_6^* \quad (25)$$

where

$$\begin{aligned}
D_1^* &= D_1^*(x, y) = D_1(x_r)^2 + D_2x_rx_z + D_3(x_z)^2 \\
D_2^* &= D_2^*(x, y) = 2D_1x_ry_r + D_2(x_ry_z + y_rx_z) + 2D_3x_zy_z \\
D_3^* &= D_1(y_r)^2 + D_2y_ry_z + D_3y_z^2 \\
D_4^* &= D_4^*(x, y) = D_1x_{rr} + D_2x_{rz} + D_3x_{zz} + D_4x_r + D_5x_z \\
D_5^* &= D_5^*(x, y) = D_1y_{rr} + D_2y_{rz} + D_3y_{zz} + D_4y_r + D_5y_z \\
D_6^* &= D_6^*(x, y) = D_6(yf(x), x, t, u, \dots)
\end{aligned}$$

and $D_i = D_i(r, z) = D_i(yf(x), x)$ for $i = 1, \dots, 5$ and

$$\begin{aligned}
x_r &= 0, \quad y_r = 1/f(z), \quad x_z = \frac{1}{b}, \quad y_z = -yf'(z)/f(z) \\
x_{rr} &= x_{rz} = x_{zz} = y_{rr} = 0, \quad y_{rz} = -f'(z)/(f(z))^2, \quad y_{zz} = -y \left(\frac{f''(z)}{f(z)} - 2 \left(\frac{f'(z)}{f(z)} \right)^2 \right)
\end{aligned}$$

Then the above coefficients reduce to

$$\begin{aligned}
D_1^*(x, y) &= \frac{D_3}{b^2} \\
D_2^*(x, y) &= \frac{D_2}{bf(z)} - 2yD_3 \frac{f'(z)}{bf(z)} \\
D_3^*(x, y) &= \frac{D_1}{(f(z))^2} - yD_2 \frac{f'(z)}{(f(z))^2} + y^2D_3 \left(\frac{f'(z)}{f(z)} \right)^2 \\
D_4^*(x, y) &= \frac{D_5}{b} \\
D_5^*(x, y) &= -D_2 \frac{f'(z)}{(f(z))^2} - yD_3 \left(\frac{f''(z)}{f(z)} - 2 \left(\frac{f'(z)}{f(z)} \right)^2 \right) + \frac{D_4}{f(z)} - yD_5 \frac{f'(z)}{f(z)} \\
D_6^*(x, y) &= D_6(yf(z), x, t, u, \dots)
\end{aligned}$$

where

$$D_i = D_i(r, z) = D_i(yf(z), x), \quad \text{for } i = 1, 2, 3, 4, 5.$$

ADI NUMERICAL METHOD

The following description of the ADI numerical method is for uniform Δx and Δy grids and of course has to be modified for unequal x and y spacing. In step 1 of the ADI numerical method the interior points to the region D' of the computational domain are labeled from left to right as illustrated in the figure 3. Assume that the system of partial differential equations to be solved

have all been normalized. Partition the segment from $x = 0$ to $x = 1$ into segments with spacing $\Delta x = 1/m_2$ so that the i th node is $i\Delta x$ and the right boundary is $m_2\Delta x$, with $0 \leq i \leq m_2$. Similarly, partition the segment from $y = 0$ to $y = 1$ into segments with spacing $\Delta y = 1/m_1$ so that the j th node is $j\Delta y$ and the top boundary is $m_1\Delta y$ with $0 \leq j \leq m_1$, where i, j, m_1 and m_2 are integers. The interior points to this grid are then labeled as illustrated in the figure 3.

Let u_n be associated with the (i, j) th node point, where

$$n = (m_2 - 1)(m_1 - 1 - j) + i \quad m_1 \text{ and } m_2 \text{ are fixed.}$$

Conversely, given the u_n point, we can determine its position i, j from the relations

$$j = m_1 - 1 - \text{Int}[(n - 1)/(m_2 - 1)]$$

$$i = n - (m_2 - 1)(m_1 - 1 - j)$$

where $\text{Int}[x]$ is the integer part of x .

u_1	u_2	u_3	u_4	\cdots	u_{m_2-1}
u_{m_2}	u_{m_2+1}	u_{m_2+2}	u_{m_2+3}	\cdots	u_{2m_2-2}
\vdots	\vdots	\vdots	\ddots	u_n	\vdots
\cdots	\cdots	\cdots	\cdots	\cdots	$u_{(m_2-1)(m_1-1)}$

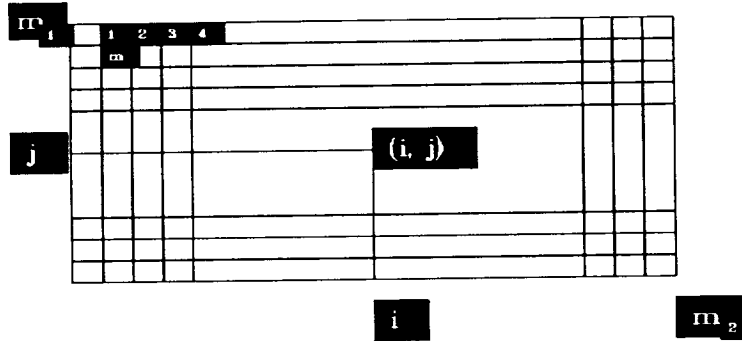


Figure 3. Step 1 labeling of interior points to domain D

Letting

$$u(i\Delta x, j\Delta y, n\Delta t) = u_{i,j}^n$$

and dropping the star notation, we then replace all partial differential equations of the form of equation (31) by difference equations having the form

$$\begin{aligned} \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = & D_1 \left(\frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{(\Delta x)^2} \right) \\ & + D_2 \left(\frac{u_{i+1,j+1}^n - u_{i+1,j-1}^n - u_{i-1,j+1}^n + u_{i-1,j-1}^n}{4\Delta x \Delta y} \right) \\ & + D_3 \left(\frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{(\Delta y)^2} \right) + D_4 \left(\frac{u_{i+1,j}^{n+1} - u_{i-1,j}^{n+1}}{2\Delta x} \right) \\ & + D_5 \left(\frac{u_{i,j+1}^n - u_{i,j-1}^n}{2\Delta y} \right) + D_6 \end{aligned}$$

which can be rearranged to the form

$$\begin{aligned} & \left(1 + \frac{2D_1\Delta t}{(\Delta x)^2} \right) u_{i,j}^{n+1} - \left(\frac{D_1\Delta t}{(\Delta x)^2} + \frac{D_4\Delta t}{2\Delta x} \right) u_{i+1,j}^{n+1} + \left(\frac{D_4\Delta t}{2\Delta x} - \frac{D_1\Delta t}{(\Delta x)^2} \right) u_{i-1,j}^{n+1} \\ & = \left(1 - \frac{2D_3\Delta t}{(\Delta y)^2} \right) u_{i,j}^n + \left(\frac{D_3\Delta t}{(\Delta y)^2} + \frac{D_5\Delta t}{2\Delta y} \right) u_{i,j+1}^n + \left(\frac{D_3\Delta t}{(\Delta y)^2} - \frac{D_5\Delta t}{2\Delta y} \right) u_{i,j-1}^n \\ & + \frac{D_2\Delta t}{4\Delta x \Delta y} \left(u_{i+1,j+1}^n - u_{i+1,j-1}^n - u_{i-1,j+1}^n + u_{i-1,j-1}^n \right) + D_6\Delta t \end{aligned}$$

Evaluating this equation at each of the interior node points gives rise to a system of $(m_2 - 1)(m_1 - 1)$ implicit linear equations which are then solved by row reduction methods.

The second step of the ADI method relabels the interior points of the computational domain from top to bottom as illustrated in the figure 4.

For this labeling we can let u_n denote the point associated with the (i, j) th node where

$$n = (m_1 - 1)(i - 1) + m_1 - j.$$

Conversely, given u_n we can solve for i and j from the relations

$$i = 1 + \text{Int}[(n - 1)/(m_1 - 1)]$$

$$j = (m_1 - 1)(i - 1) + m_1 - n$$

$$\begin{array}{cccccc}
u_1 & u_{m_1} & \cdots & \cdots & \cdots & \\
u_2 & u_{m_1+1} & \cdots & \cdots & \cdots & \\
u_3 & u_{m_1+2} & \cdots & \cdots & \cdots & \\
u_4 & u_{m_1+3} & \ddots & \vdots & \vdots & \\
\vdots & \cdots & \cdots & u_n & \vdots & \\
u_{m_1-1} & u_{2m_1-2} & \cdots & \cdots & \cdots &
\end{array}$$

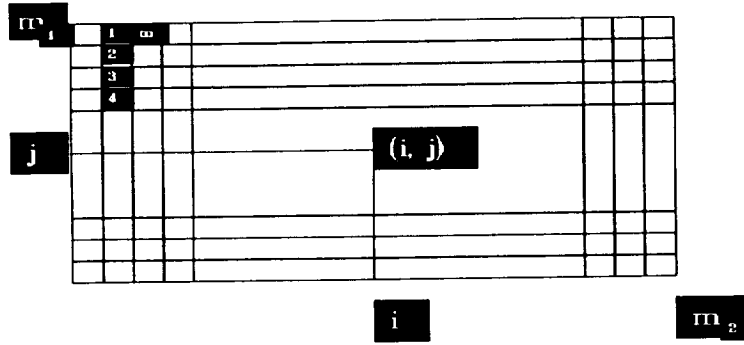


Figure 4. Step 2 labeling of interior points to domain D'

For step 2, all the partial differential equations of the form of equation (31) are replaced by the difference equations having the form

$$\begin{aligned}
\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = & D_1 \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{(\Delta x)^2} \right) \\
& + D_2 \left(\frac{u_{i+1,j+1}^n - u_{i+1,j-1}^n - u_{i-1,j+1}^n + u_{i-1,j-1}^n}{4\Delta x \Delta y} \right) \\
& + D_3 \left(\frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{(\Delta y)^2} \right) + D_4 \left(\frac{u_{i+1,j}^n - u_{i-1,j}^n}{2\Delta x} \right) \\
& + D_5 \left(\frac{u_{i,j+1}^{n+1} - u_{i,j-1}^{n+1}}{2\Delta y} \right) + D_6
\end{aligned}$$

which can be rearranged to the form

$$\begin{aligned}
& \left(1 + \frac{2D_3\Delta t}{(\Delta y)^2}\right) u_{i,j}^{n+1} - \left(\frac{D_3\Delta t}{(\Delta y)^2} + \frac{D_5\Delta t}{2\Delta y}\right) u_{i,j+1}^{n+1} + \left(\frac{D_5\Delta t}{2\Delta y} - \frac{D_3\Delta t}{(\Delta y)^2}\right) u_{i,j-1}^{n+1} \\
& = \left(1 - \frac{2D_1\Delta t}{(\Delta x)^2}\right) u_{i,j}^n + \left(\frac{D_1\Delta t}{(\Delta x)^2} + \frac{D_4\Delta t}{2\Delta x}\right) u_{i+1,j}^n + \left(\frac{D_1\Delta t}{(\Delta x)^2} - \frac{D_4\Delta t}{2\Delta x}\right) u_{i-1,j}^n \\
& + \frac{D_2\Delta t}{4\Delta x\Delta y} \left(u_{i+1,j+1}^n - u_{i+1,j-1}^n - u_{i-1,j+1}^n + u_{i-1,j-1}^n\right) + D_6\Delta t
\end{aligned}$$

Applying this difference equation to each interior node point results in an implicit system of $(m_1 - 1)(m_2 - 1)$ simultaneous linear equations which must be solved for the values of u at the node points.

One can see that the finer the interior grid there results a larger system of linear equations to be solved. Also the results from the ADI numerical method are more accurate on the even numbered time steps. Additional complications results when employing the unequal step size approximations illustrated in the figure 5. The unequal step sizes are necessary to handle large gradients occurring in any of the dependent variables. The computational region is therefore divided into 6 regions as illustrated in the figure 5. The density of node points can be changed in each region by selecting different step sizes in the computational coordinates.

1	2	3
4	5	6

Figure 5. Unequal x and y spacing.

In the case of unequal grid spacing we employ the difference approximations.

$$\begin{aligned}
\frac{\partial u}{\partial x} &\approx \frac{1}{\Delta x_1 + \Delta x_2} \left[\frac{\Delta x_2}{\Delta x_1} (u(x + \Delta x_1, y) - u(x, y)) - \frac{\Delta x_1}{\Delta x_2} (u(x - \Delta x_2, y) - u(x, y)) \right] \\
\frac{\partial u}{\partial y} &\approx \frac{1}{\Delta y_1 + \Delta y_2} \left[\frac{\Delta y_2}{\Delta y_1} (u(x, y + \Delta y_1) - u(x, y)) - \frac{\Delta y_1}{\Delta y_2} (u(x, y - \Delta y_2) - u(x, y)) \right] \\
\frac{\partial^2 u}{\partial x^2} &\approx \frac{2}{\Delta x_1 + \Delta x_2} \left[\frac{u(x + \Delta x_1, y) - u(x, y)}{\Delta x_1} + \frac{u(x - \Delta x_2, y) - u(x, y)}{\Delta x_2} \right] \\
\frac{\partial^2 u}{\partial y^2} &\approx \frac{2}{\Delta y_1 + \Delta y_2} \left[\frac{u(x, y + \Delta y_1) - u(x, y)}{\Delta y_1} + \frac{u(x, y - \Delta y_2) - u(x, y)}{\Delta y_2} \right] \\
\frac{\partial^2 u}{\partial x \partial y} &\approx \frac{u(x + \Delta x_1, y + \Delta y_1) - u(x + \Delta x_1, y - \Delta y_2)}{(\Delta x_1 + \Delta x_2)(\Delta y_1 + \Delta y_2)} \\
&\quad + \frac{u(x - \Delta x_2, y - \Delta y_2) - u(x - \Delta x_2, y + \Delta y_1)}{(\Delta x_1 + \Delta x_2)(\Delta y_1 + \Delta y_2)}
\end{aligned}$$

SPECIAL CASE-ELECTRIC FIELD IN VACUUM

In a vacuum we solve

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (26)$$

over the domain $0 \leq r \leq f(z)$ and $0 \leq z \leq 1$. Using the transformation equations

$$z = x \quad r = yf(z)$$

where

$$f(z) = .2 + x \tan(8\pi/180)$$

is used to describe a straight line nozzle boundary. The equation (26) then transforms to

$$\frac{\partial^2 \phi}{\partial x^2} + a(x, y) \frac{\partial^2 \phi}{\partial x \partial y} + b(x, y) \frac{\partial^2 \phi}{\partial y^2} + c(x, y) \frac{\partial \phi}{\partial y} = 0 \quad (27)$$

over the domain $0 \leq x \leq 1, 0 \leq y \leq 1$ where

$$\begin{aligned}
a(x, y) &= -2y \frac{f'(z)}{f(z)} \\
b(x, y) &= \frac{1}{f^2(z)} + \left(y \frac{f'(z)}{f(z)} \right)^2 \\
c(x, y) &= \frac{1}{yf^2(z)} - y \left(\frac{f''(z)}{f(z)} - 2 \left(\frac{f'(z)}{f(z)} \right)^2 \right)
\end{aligned}$$

FIGURE 6. POTENTIAL FIELD IN VACUUM

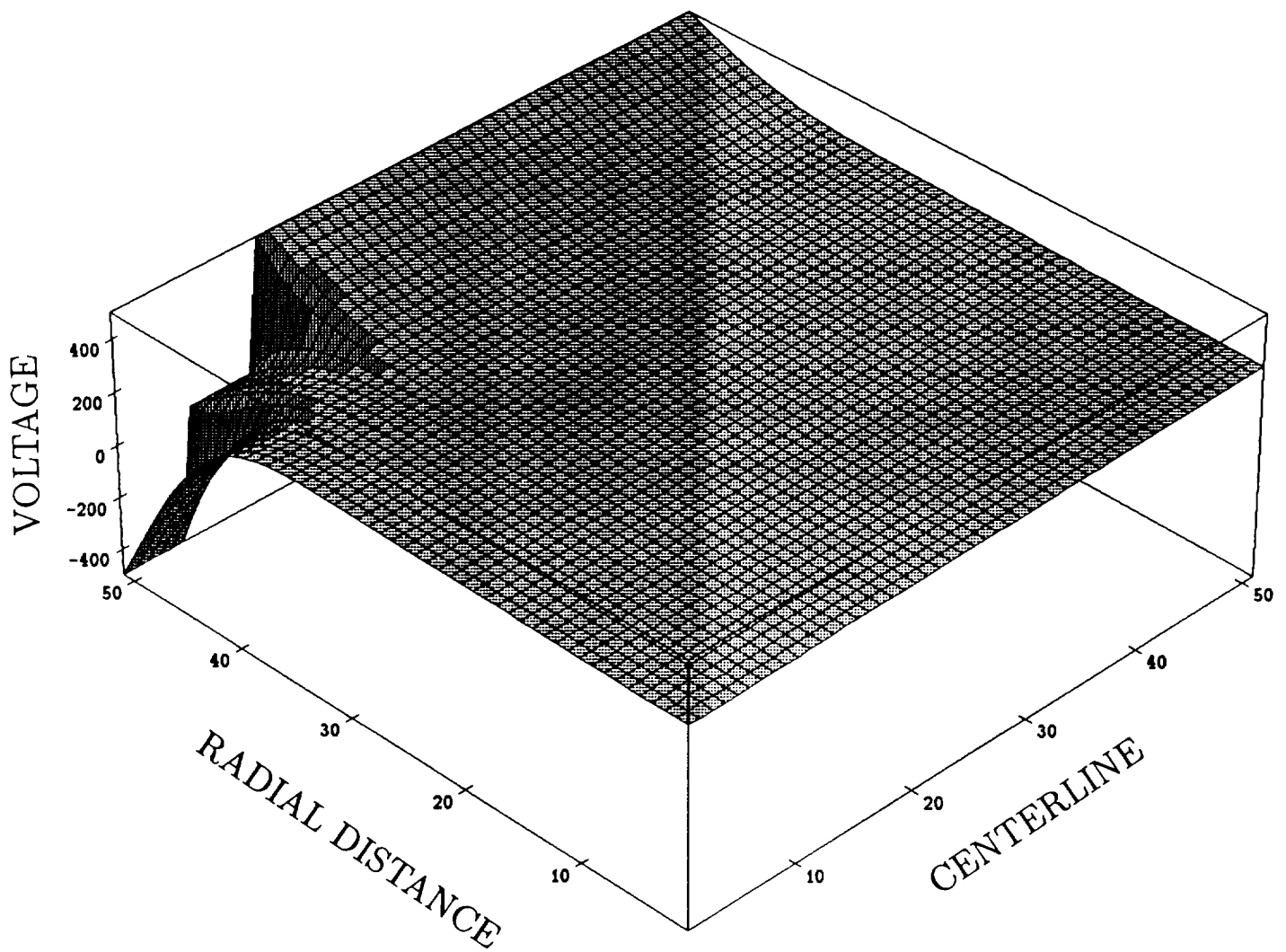


FIGURE 7.

MAGNITUDE OF ELECTRIC FIELD IN VACUUM

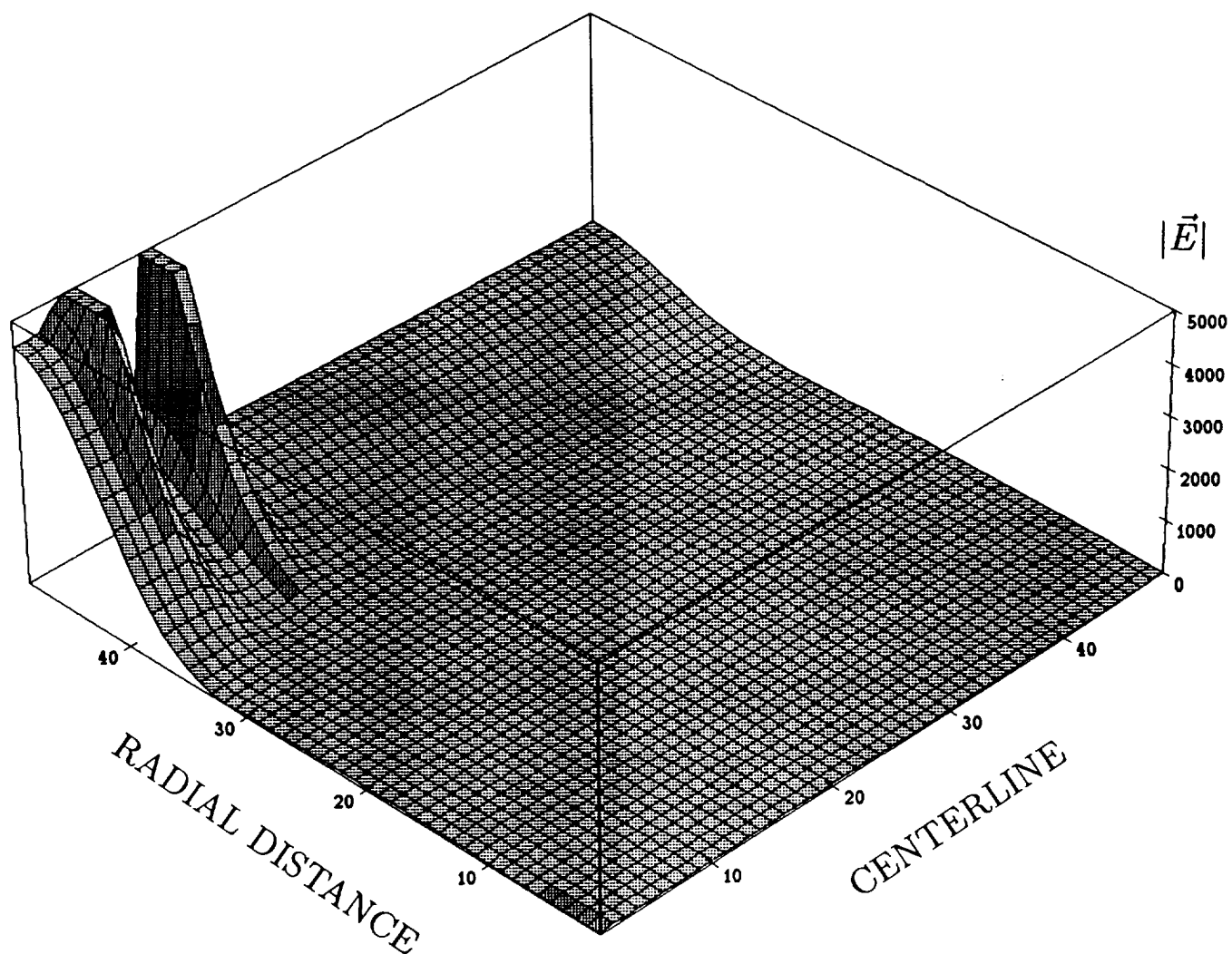


FIGURE 8. POTENTIAL FIELD IN VACUUM

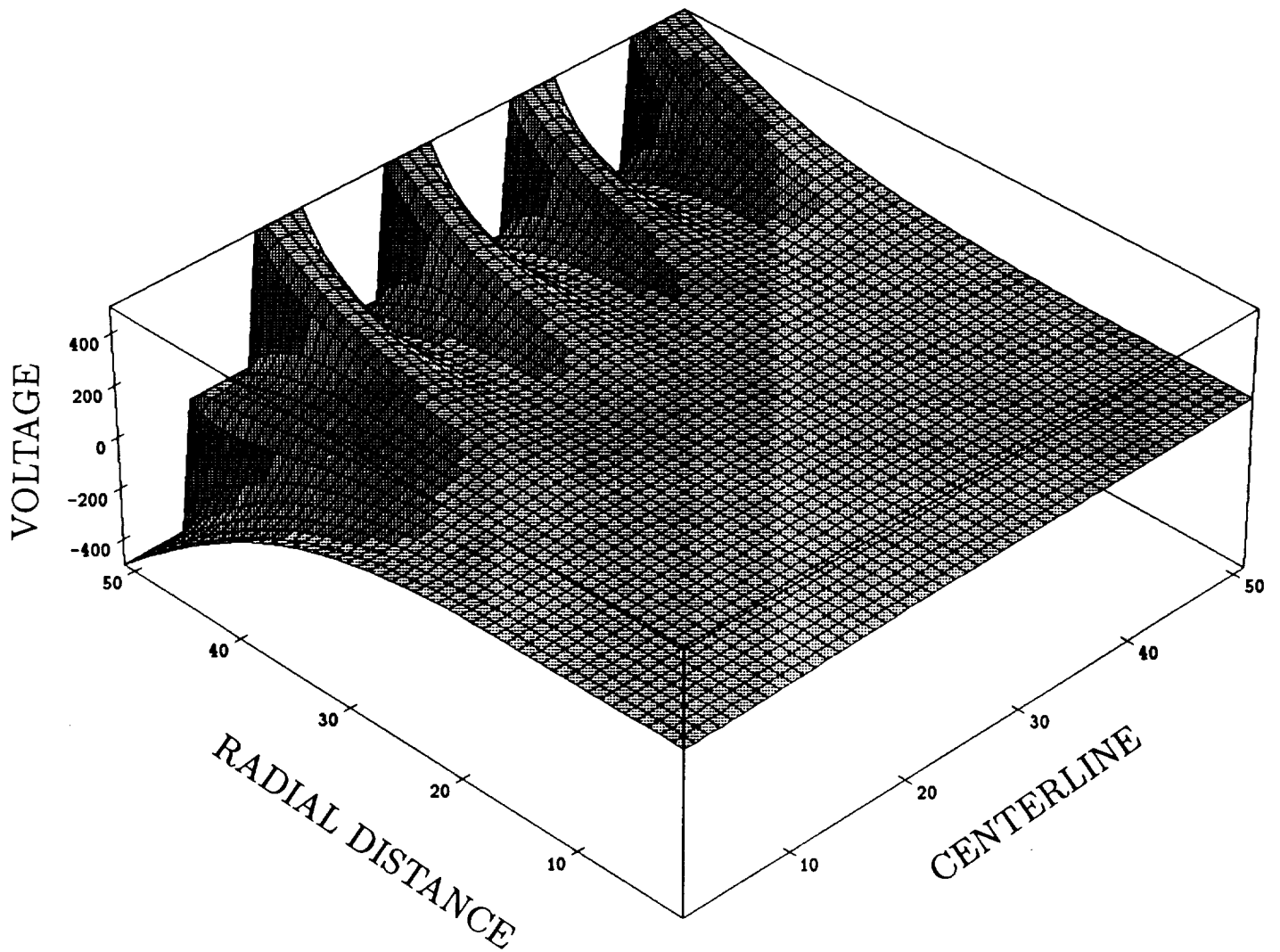
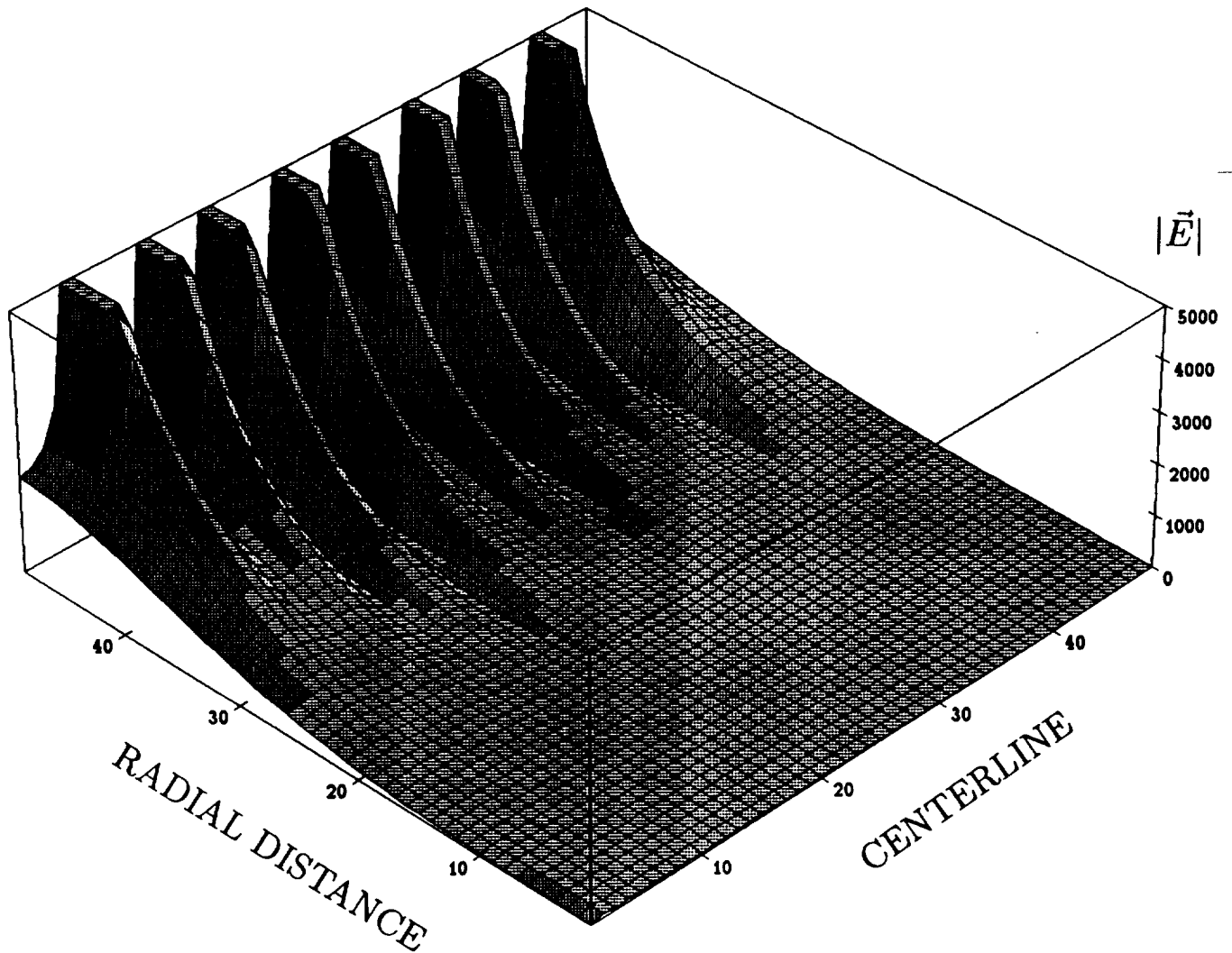


FIGURE 9.
MAGNITUDE OF ELECTRIC FIELD IN VACUUM



Using the notation $u_{i,j} = u(i\Delta x, j\Delta y)$ and the difference approximations

$$\begin{aligned} u_{xx} &\approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \\ u_{xy} &\approx \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4h^2} \\ u_{yy} &\approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} \\ u_y &\approx \frac{u_{i+1,j} - u_{i-1,j}}{2h} \end{aligned}$$

and defining

$$d(x, y) = \frac{2}{h^2}(1 + b(x, y))$$

the equation (26) reduces to the difference equation

$$\begin{aligned} u_{i,j} = \frac{1}{d_{ij}} \left\{ \frac{u_{i+1,j} + u_{i-1,j}}{h^2} + \frac{a_{ij}}{4h^2} (u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}) \right. \\ \left. + \frac{b_{ij}}{h^2} (u_{i,j+1} + u_{i,j-1}) + \frac{c_{ij}}{2h} (u_{i+1,j} - u_{i-1,j}) \right\} \end{aligned}$$

This difference equation is subject to the boundary conditions

$$u_{i,j_{max}} = \text{assigned potential value}$$

$$u_{0,j} = u_{2,j}$$

$$u_{i_{max},j} = u_{i_{max}-2,j}$$

$$u_{i,0} = u_{i,2}$$

which represent zero derivative boundary conditions along the other three sides. The figures 6,7,8 and 9 illustrate the potential function for two different nozzle configurations where the cathode is assigned a value of -500 volts and the anode(s) is assigned a value of $+500$ volts.

FLOW AND HEAT TRANSFER THROUGH A POROUS MEDIA

In the figure 10 a porous material is heated with 40kw of power from a solar simulator. We assume that the solid porous material is heated to a uniform temperature T_s and that a gas flows through the porous material and is heated.

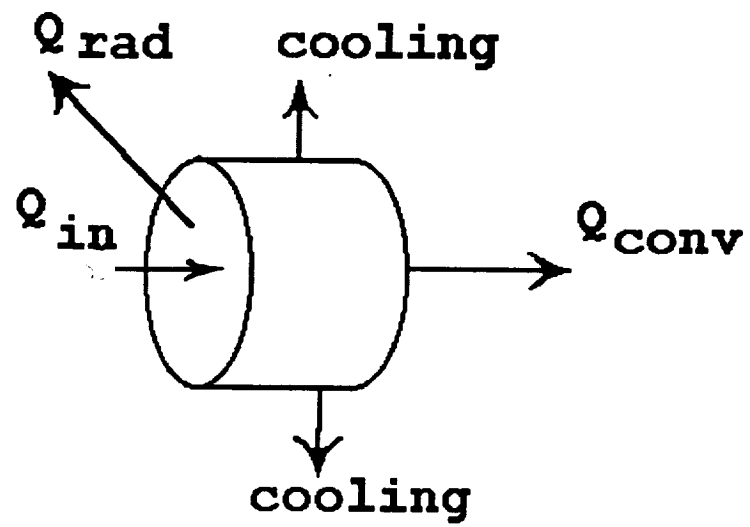


Figure 10. Heat transfer through porous material.

In the following discussions we use the notations:

$$\phi = \text{porosity} \quad 0 < \phi < 1$$

$$T_g, T_s = \text{Temperature of gas and solid} \quad (K)$$

$$\rho_g, \rho_s = \text{Density of gas and solid} \quad (gm/cm^3)$$

$$K_g, K_s = \text{Thermal conductivity of gas and solid} \quad (cal/s\,cm^2\,K/cm)$$

$$K_e = (1 - \phi)K_s + \phi K_g = \text{Effective thermal conductivity}$$

$$u = \text{Velocity of gas} \quad (cm/sec)$$

$$\alpha_g = K_g / C_{pg} \rho_g = \text{Thermal diffusivity of gas} \quad (cm^2/sec)$$

$$L = \text{Thickness of porous material} \quad (cm)$$

$$Pe = \frac{uL}{\alpha_g} = \text{Peclet number} \quad (\text{dimensionless})$$

$$r = \text{Radius of disk} \quad (cm)$$

$$h = \text{Heat transfer coefficient} \quad (cal/s\,cm^2\,K)$$

$$Q_0 = \text{Input power} \quad (Kw)$$

$$A = \text{Surface area of disk} \quad (cm^2)$$

$$C_{pg}, C_{ps} = \text{Specific heat of gas and solid} \quad (cal/gmK)$$

$$R_s = \text{Ratio of surface area to volume of porous media} \quad (cm^2/cm^3)$$

$$U = T_s / T_{g0} = \text{Dimensionless temperature ratio}$$

$$V = T_g / T_{g0} = \text{Dimensionless temperature ratio}$$

$$\tau = \frac{tu}{L} = \text{Dimensionless time}$$

$$X = \frac{x}{L} = \text{Dimensionless distance}$$

Following reference 1, the basic equations describing the heat transfer to a gas moving through a porous media are given by

$$\rho_g C_{pg} \left(\frac{\partial T_g}{\partial t} + u \frac{\partial T_g}{\partial x} \right) = K_g \frac{\partial^2 T_g}{\partial x^2} - hR(T_g - T_s) \quad (28)$$

$$\rho_s C_{ps} \frac{\partial T_s}{\partial t} = K_s \frac{\partial^2 T_s}{\partial x^2} - hR(T_s - T_g) \quad (29)$$

for $0 \leq x \leq L$ and $t \geq 0$.

The equations (1) and (2) are subject to the boundary conditions

$$K_s \frac{\partial T_s}{\partial z} \Big|_{z=0} = \frac{239 Q_0}{A} - \epsilon \sigma (T_s^4 - T_{g0}^4) - \rho_g C_{pg} u (T_s - T_{g0}) \quad (30)$$

and

$$-K_s \frac{\partial T_s}{\partial r} = h(T_s - T_w) \quad (31)$$

where all terms have been scaled to the units of $cal/s\,cm^2$ and

$$\sigma = 5.67(10^{-12})(.239) \, cal/s\,cm^2\,K^4$$

is the Stephan-Boltzman constant, ϵ is the emissivity. In addition we have the boundary conditions

$$\rho_g C_{pg} u \frac{\partial T_g}{\partial z} \Big|_{z=0} = hR(T_s - T_g) \quad (32)$$

$$\frac{\partial T_s}{\partial z} \Big|_{z=L} = 0 \quad \text{and} \quad \frac{\partial T_g}{\partial z} \Big|_{z=L} = 0 \quad (33)$$

We further assume that the initial conditions are

$$T_s = T_g = T_{g0}. \quad (34)$$

Introducing the dimensionless variables U, V, X, τ , the above equations can be written

$$\frac{\partial U}{\partial \tau} = A_0 \left(\frac{1}{r_0^2} \frac{\partial^2 U}{\partial R^2} + \frac{1}{r_0^2 R} \frac{\partial U}{\partial R} + \frac{1}{L^2} \frac{\partial^2 U}{\partial Z^2} \right) - B_0(U - V) \quad (35)$$

$$\frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial Z} = A_1 \left(\frac{1}{r_0^2} \frac{\partial^2 V}{\partial R^2} + \frac{1}{r_0^2 R} \frac{\partial V}{\partial R} + \frac{1}{L^2} \frac{\partial^2 V}{\partial Z^2} \right) - B_1(V - U) \quad (36)$$

$$\frac{\partial U}{\partial R} \Big|_{R=1} = \frac{-hr}{K_s}(U - 1.) \quad (37)$$

where

$$A_1 = \frac{L^2}{Pe} \quad B_1 = \frac{hR_s L^2}{Pe K_g} \quad A_0 = \frac{K_s L^2}{\alpha_g \rho_s C_{ps} Pe} \quad B_0 = \frac{hR_s L^2}{\alpha_g \rho_s C_{ps} Pe}$$

The equations (35) and (36) are subject to the boundary conditions

$$\frac{\partial U}{\partial Z} \Big|_{Z=0} = A_3 - B_3(U^4 - V^4) - C_3(U - V) \quad (38)$$

$$\frac{\partial V}{\partial Z} \Big|_{Z=0} = B_4(U - V) \quad (39)$$

$$\frac{\partial U}{\partial Z} \Big|_{Z=1} = 0, \quad \frac{\partial V}{\partial Z} \Big|_{Z=1} = 0 \quad (40)$$

where

$$A_3 = \frac{239Q(r_0 R)L}{AK_e T_{g0}}, \quad B_3 = \frac{\epsilon \sigma T_{g0}^3 L}{K_e}, \quad C_3 = \frac{K_g Pe}{K_e}, \quad B_4 = \frac{h R_s L^2}{\alpha_g Pe \rho_g C_{pg}} \quad (41)$$

The initial conditions are

$$U(0, t) = 1 \quad \text{and} \quad V(0, t) = 1 \quad (42)$$

We divide the intervals $0 \leq Z \leq 1$ and $0 < R < 1$ into N parts with step sizes $\Delta Z = \Delta R = 1/N$. We desire to represent the temperature of the gas and solid at the positions $R = i\Delta R$ and $Z = j\Delta Z$ for the time $\tau = n\Delta\tau$ which is based upon the given temperatures at time τ . Let $U(i\Delta R, j\Delta Z, n\Delta\tau) = U_{i,j}^n$ and $V(i\Delta R, j\Delta Z, n\Delta\tau) = V_{i,j}^n$, and use difference approximations to write the above equations as difference equations. We use the ADI (Alternating Direction Implicit) method to solve the above system of coupled partial differential equations.

Material Properties

From reference 15, we obtained the following empirical data for Hafnium carbide.

Temperature deg K	Thermal Conductivity of Hfc W/cm K
560	0.09
800	0.12
1100	0.13
2000	0.15
2500	0.25
3000	0.29

The best fit second degree polynomial to the above data is given by

$$K_s(T) = .0361887 + 1.03093(10^{-4}) T - 1.2077(10^{-8}) T^2$$

Temperature deg K	Specific Heat of Hfc cal/g K
60	0.011
95	0.020
300	0.045
600	0.065
1000	0.065
3000	0.065

Table lookup will be used to fit the Specific Heat data.

Temperature deg K	Emittance
300	1.00
1100	0.98
1900	0.90
2500	0.70
2900	0.62

The above data is represented by the approximating function

$$\epsilon(T) = .8 - .2 \tanh((T - 2100)/1000).$$

We use the above data to construct empirical relations to represent the constants in the above system of coupled partial differential equations. The Appendix A contains graphical output from the computer analysis of the heat transfer in a porous media.

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APPENDIX A

GRAPHICAL REPRESENTATION OF RESULTS

Solution of the coupled equations describing flow of gas through a porous media. The input power in Kw is assumed to be a cosine curve of the form

$$Q(R) = 40. \cos\left(\frac{\pi R}{2}\right).$$

Solution of coupled equations is by ADI (Alternating Direction Implicit) technique. All dimensions have been normalized. The radial and axial directions range from 0 to 1. The temperatures of the gas (V) and solid (U) have been normalized by the equations

$$U = TS/TG0 \qquad V = TG/TG0$$

where $TG0 = 300$ K.

The figures A1 through A16 illustrate the temperature change for the gas and solid as a function of the normalized time

$$\tau = \frac{t u}{L}$$

where t is real time, u is velocity, and L is length in the axial direction.

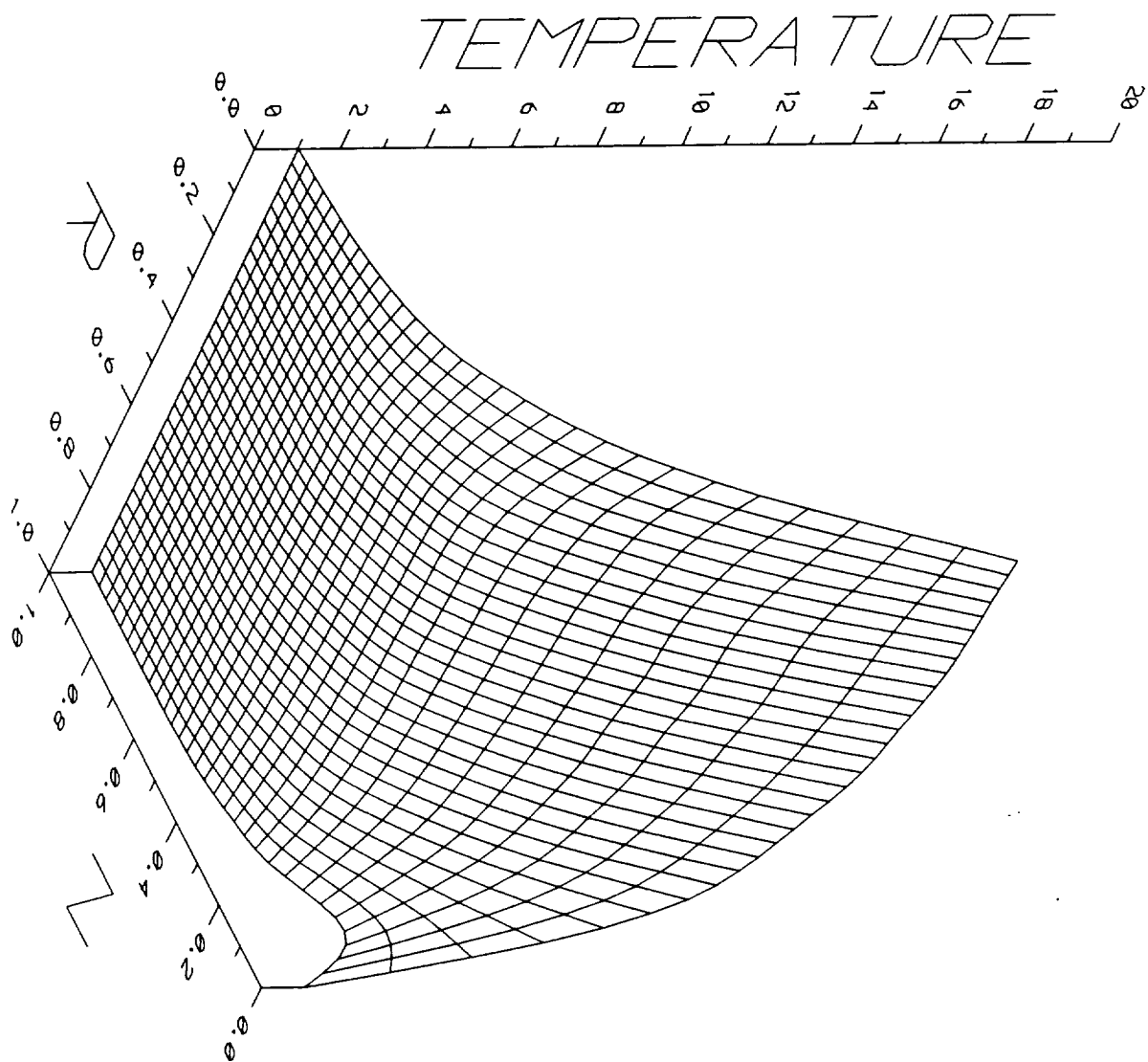


Figure A1. Normalized Gas Temperature at $\tau = 1.0$

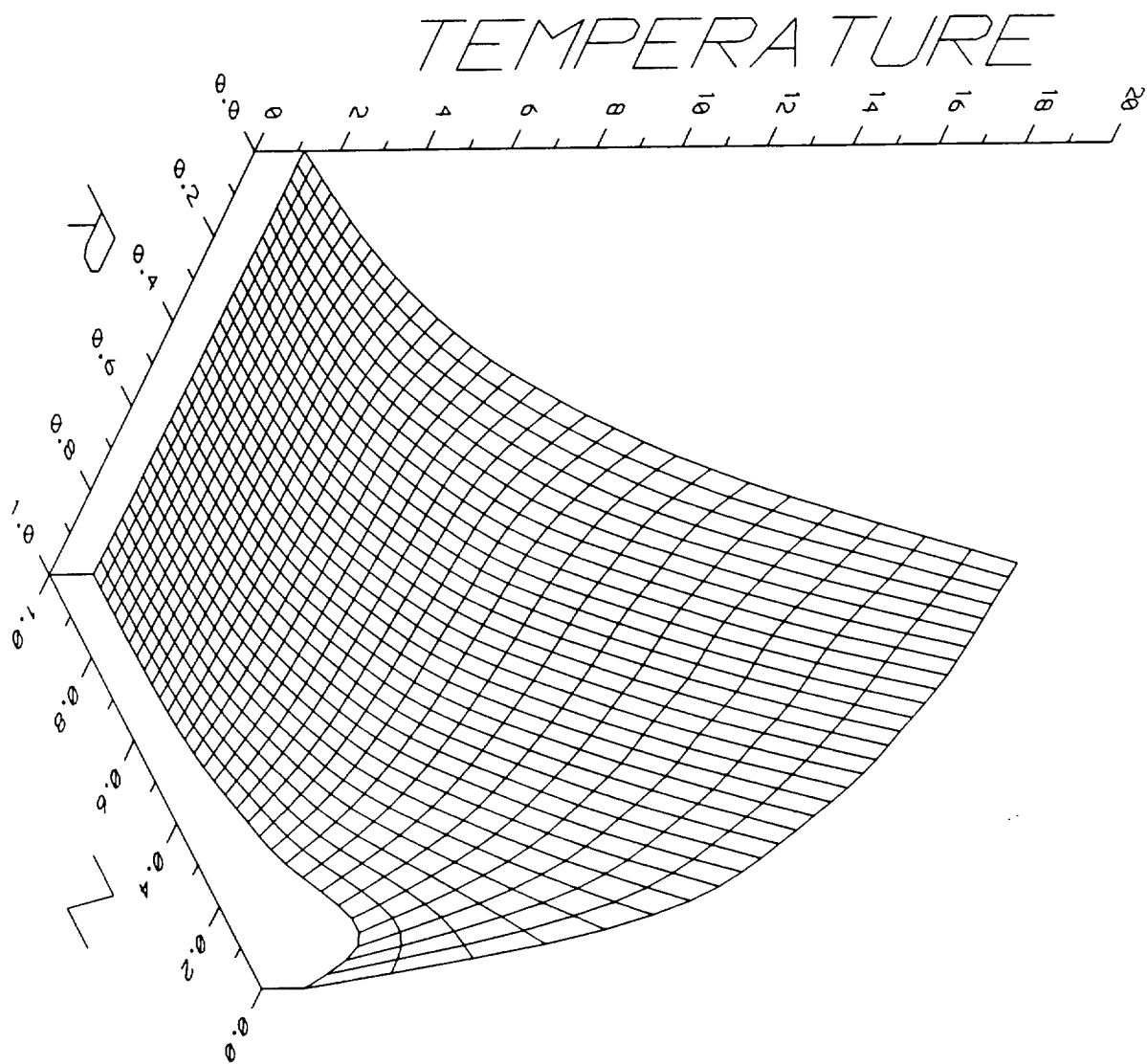


Figure A2. Normalized Gas Temperature at $\tau = 2.0$

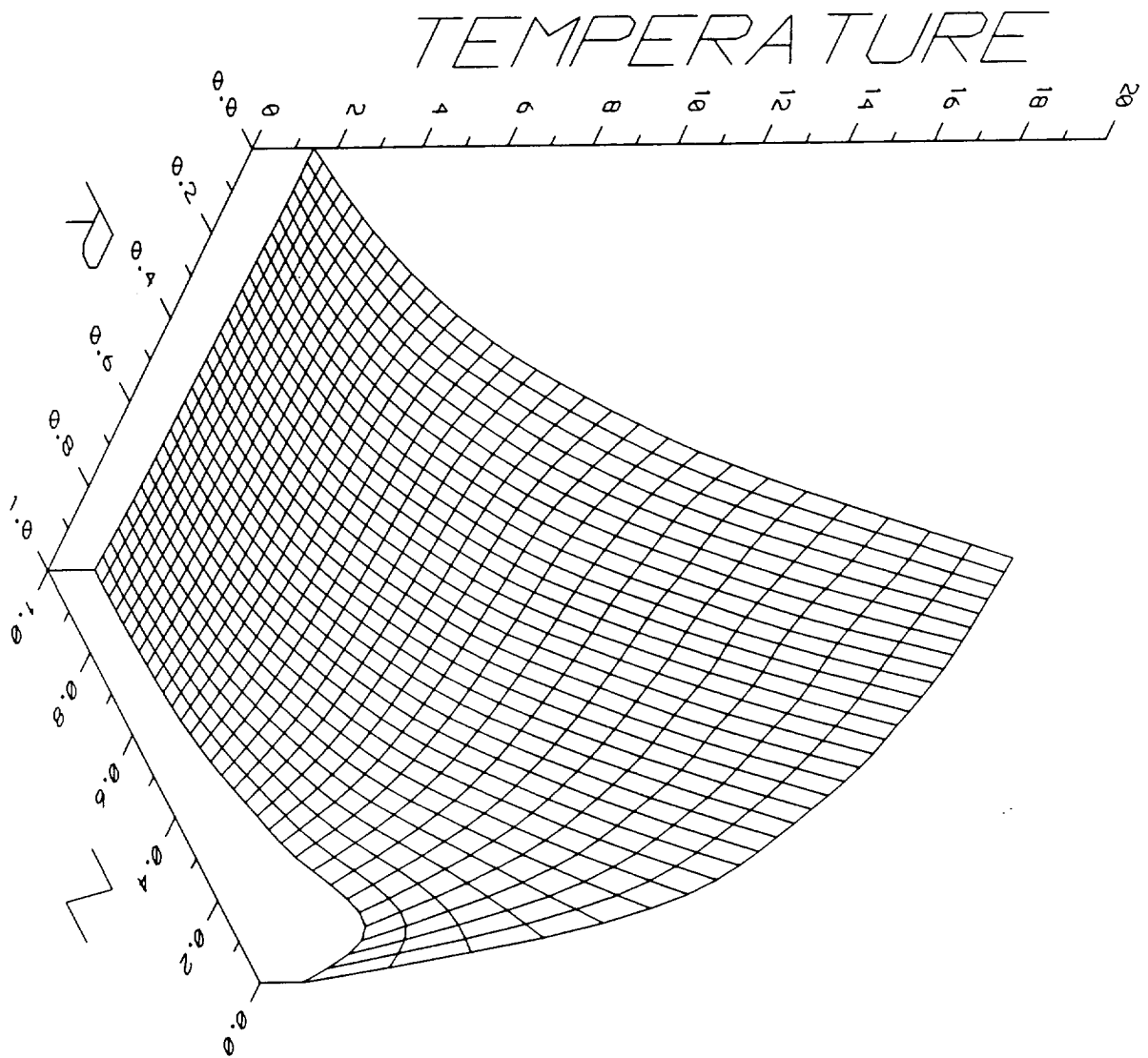


Figure A3. Normalized Gas Temperature at $\tau = 3.0$

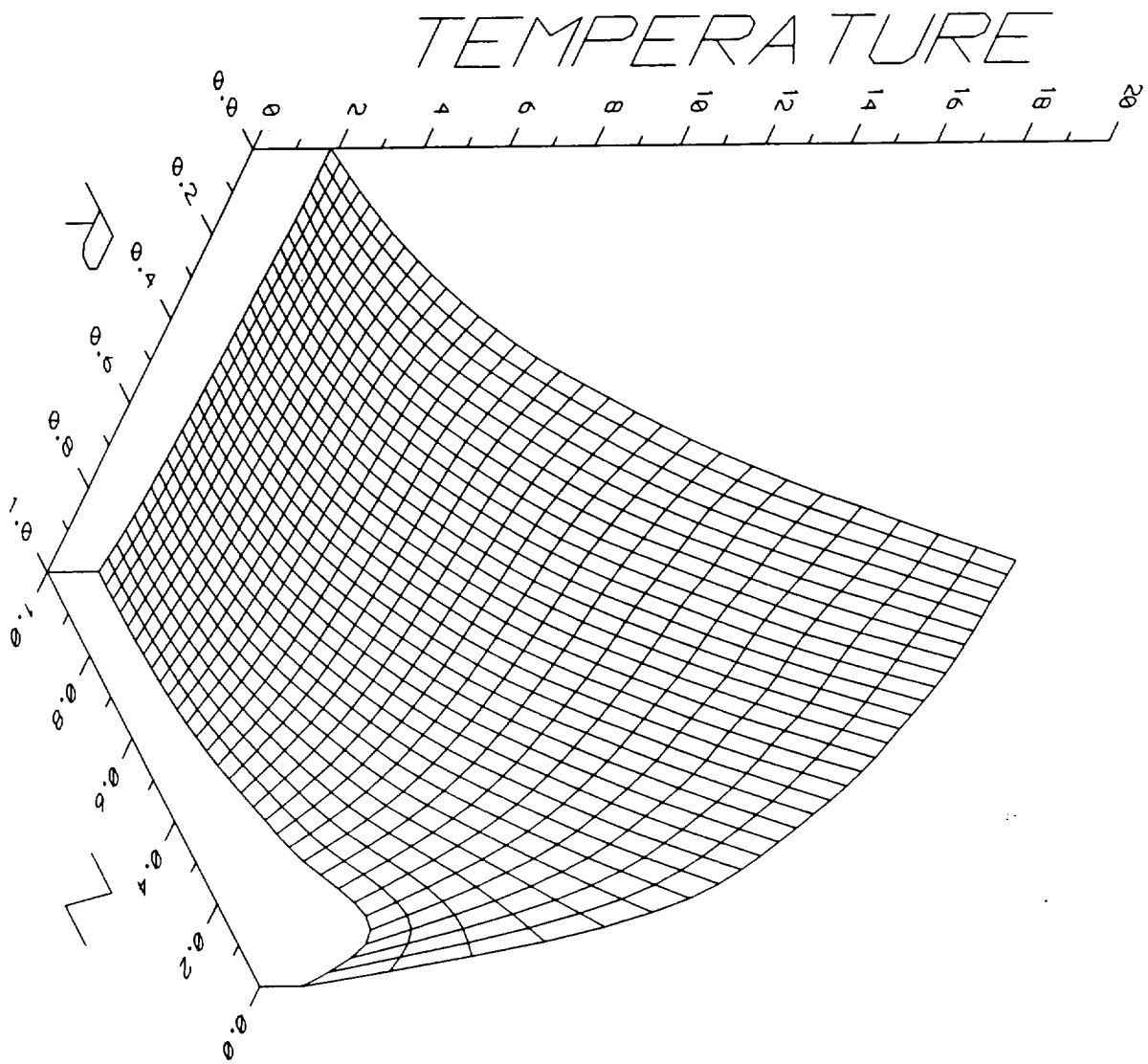


Figure A4. Normalized Gas Temperature at $\tau = 4.0$

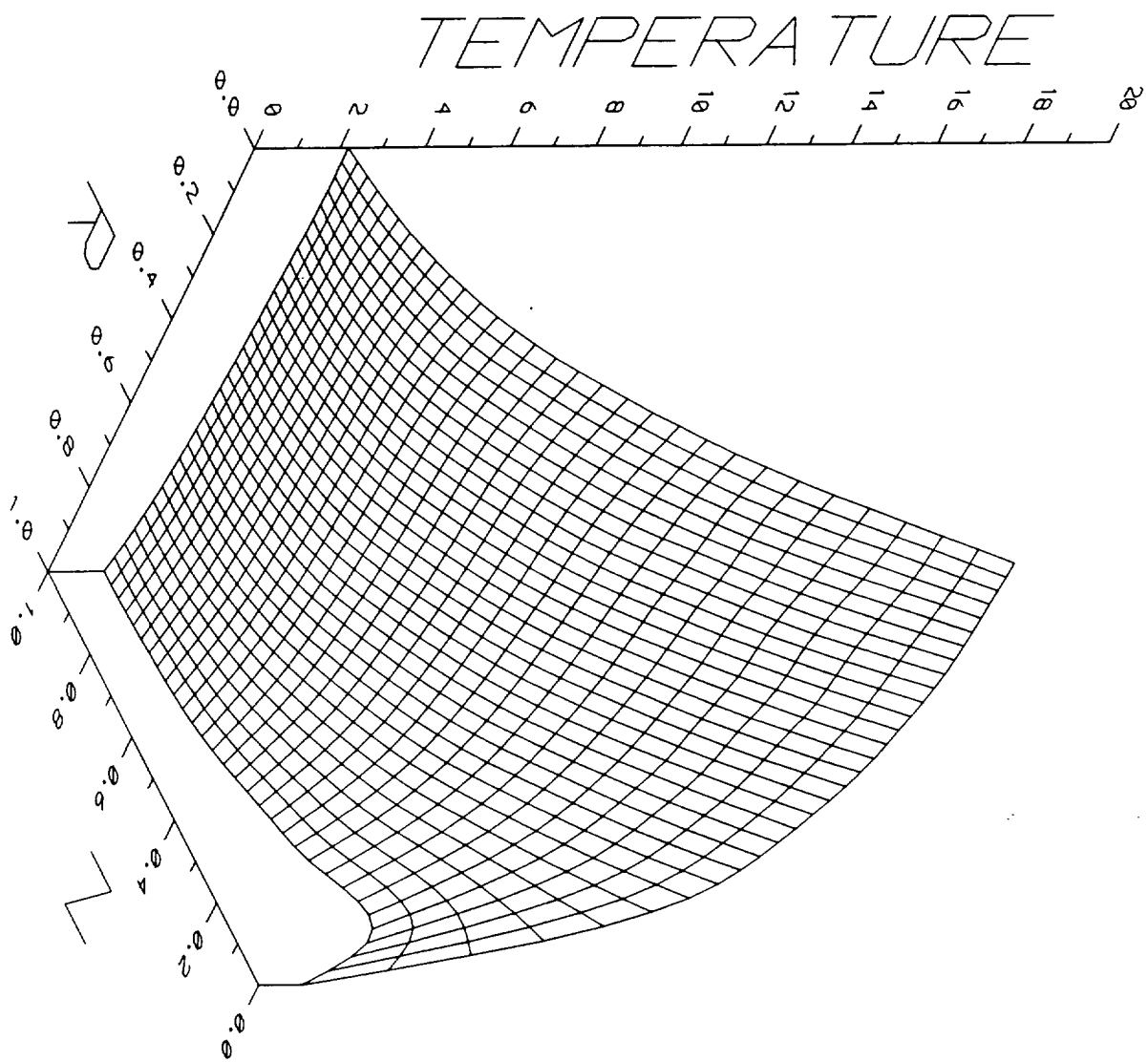


Figure A5. Normalized Gas Temperature at $\tau = 5.0$

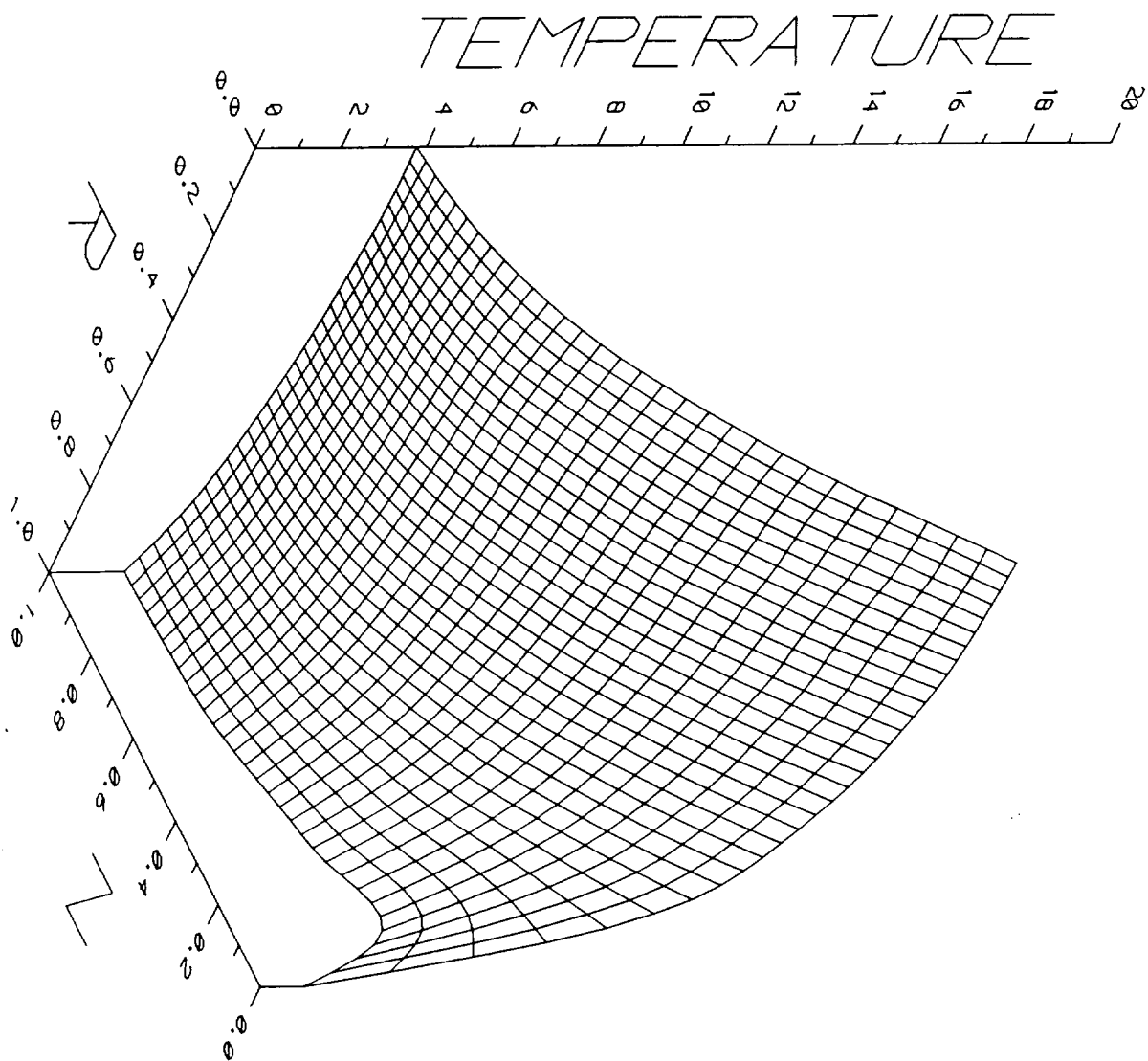


Figure A6. Normalized Gas Temperature at $\tau = 10.0$

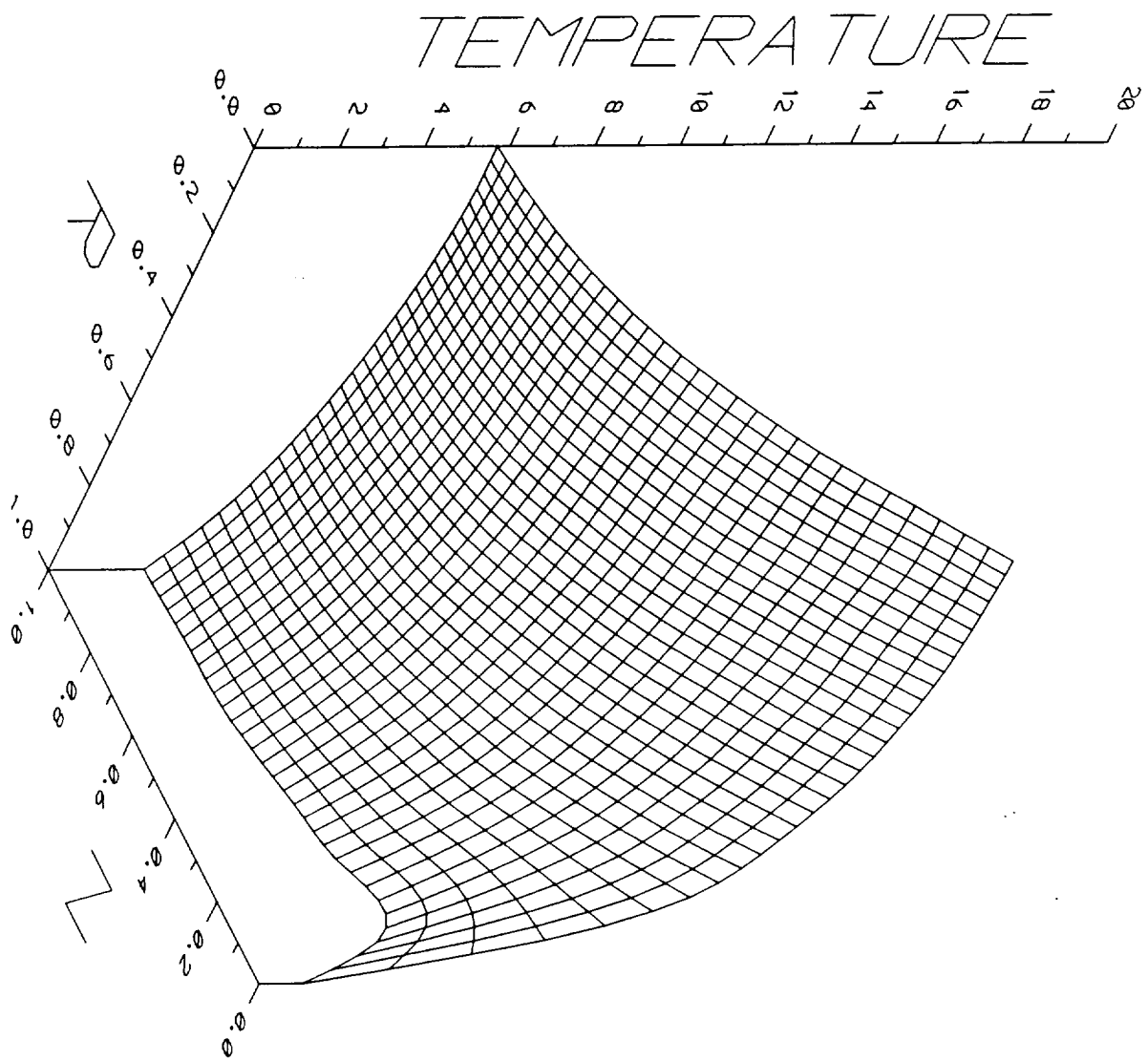


Figure A7. Normalized Gas Temperature at $\tau = 20.0$

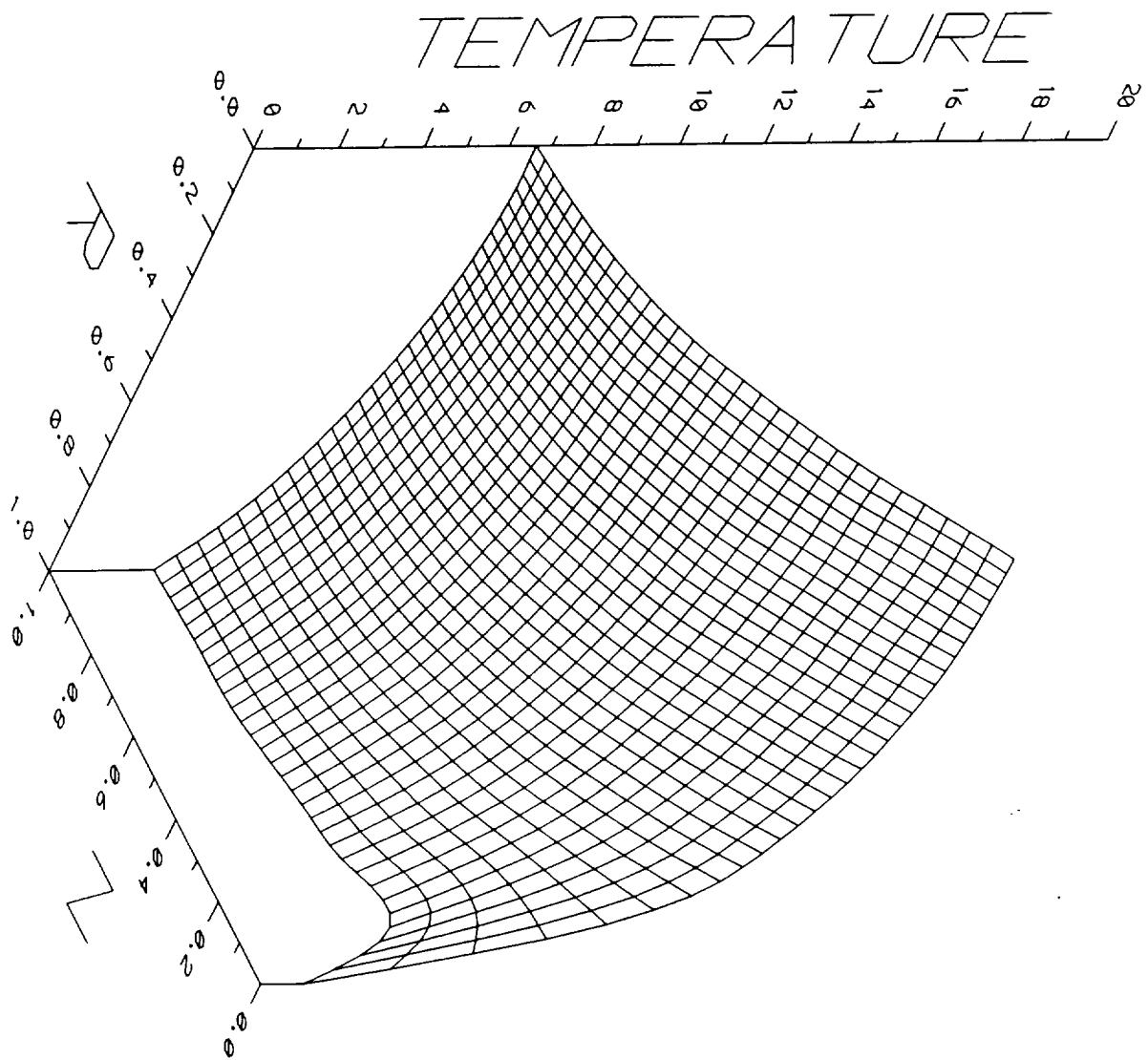


Figure A8. Normalized Gas Temperature at $\tau = 30.0$

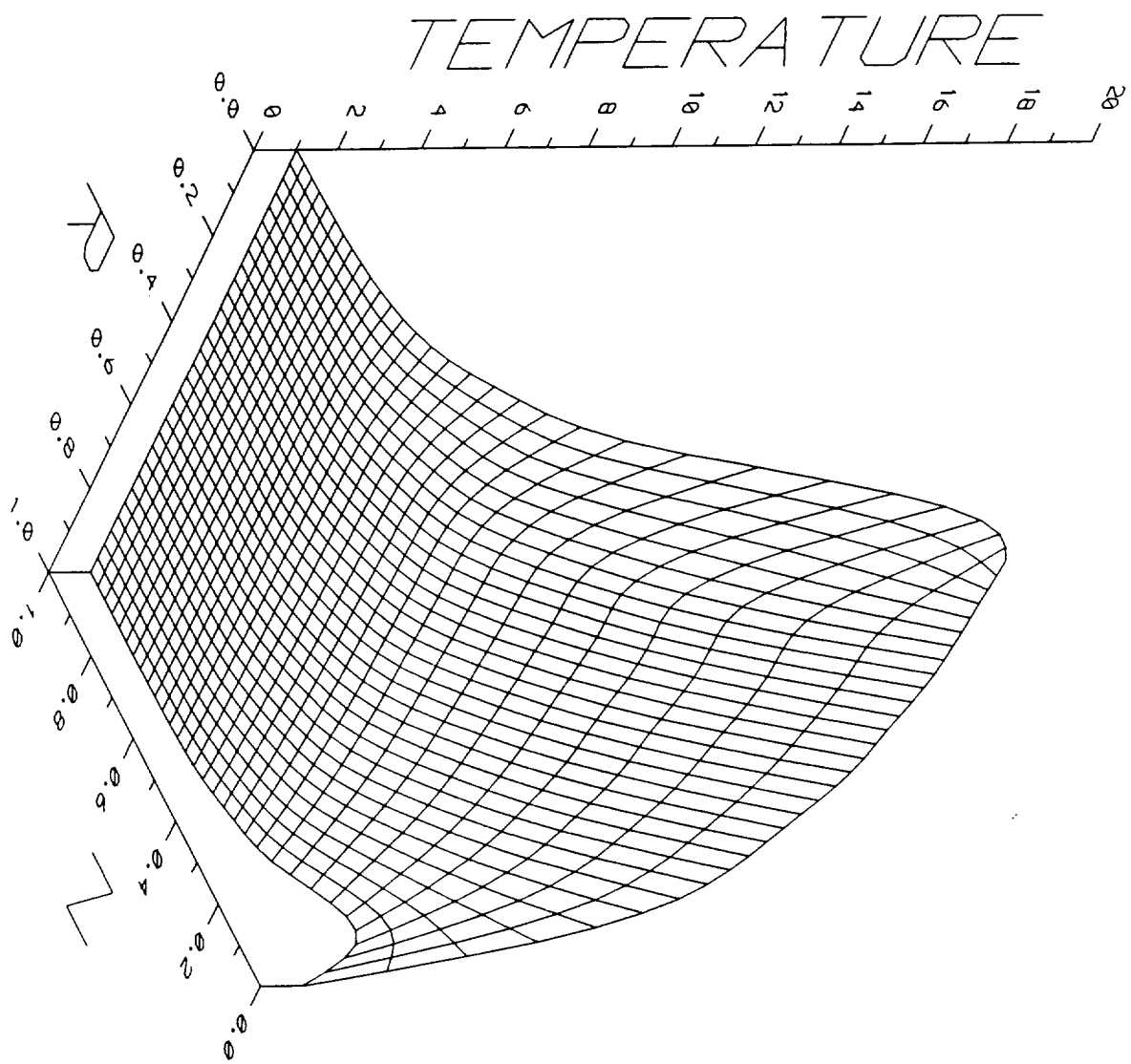


Figure A9. Normalized Solid Temperature at $\tau = 1.0$

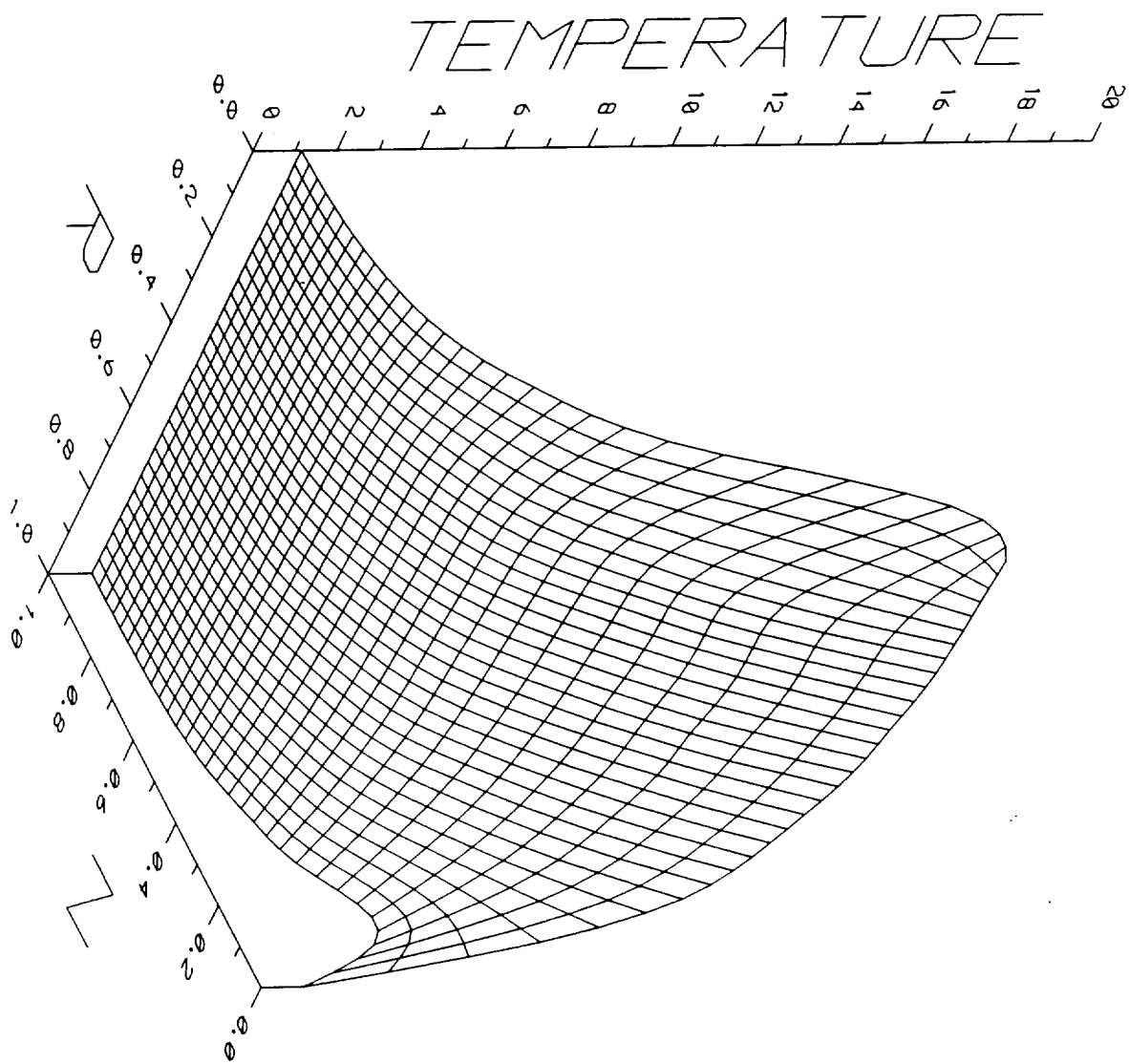


Figure A10. Normalized Solid Temperature at $\tau = 2.0$

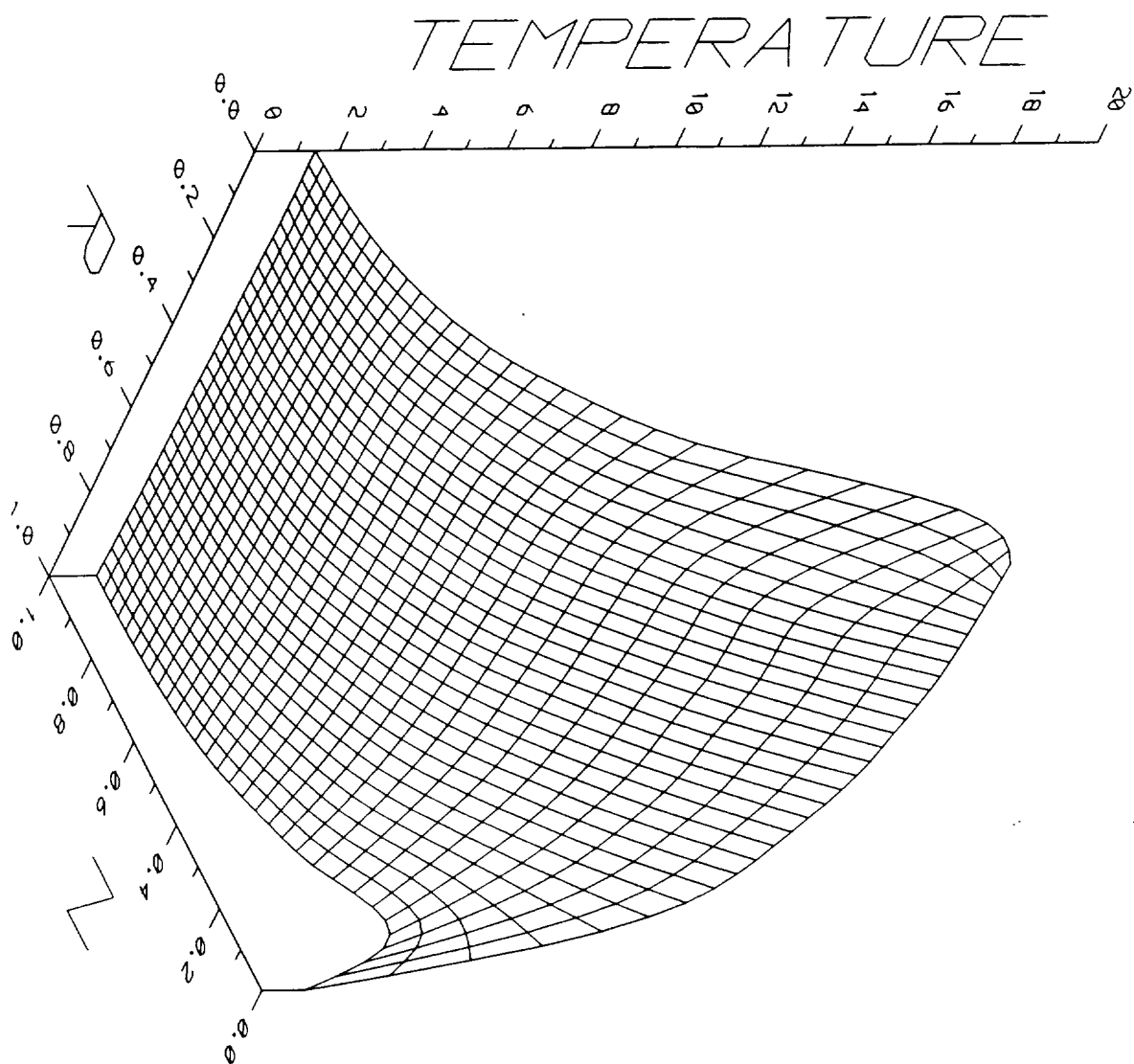


Figure A11. Normalized Solid Temperature at $\tau = 3.0$

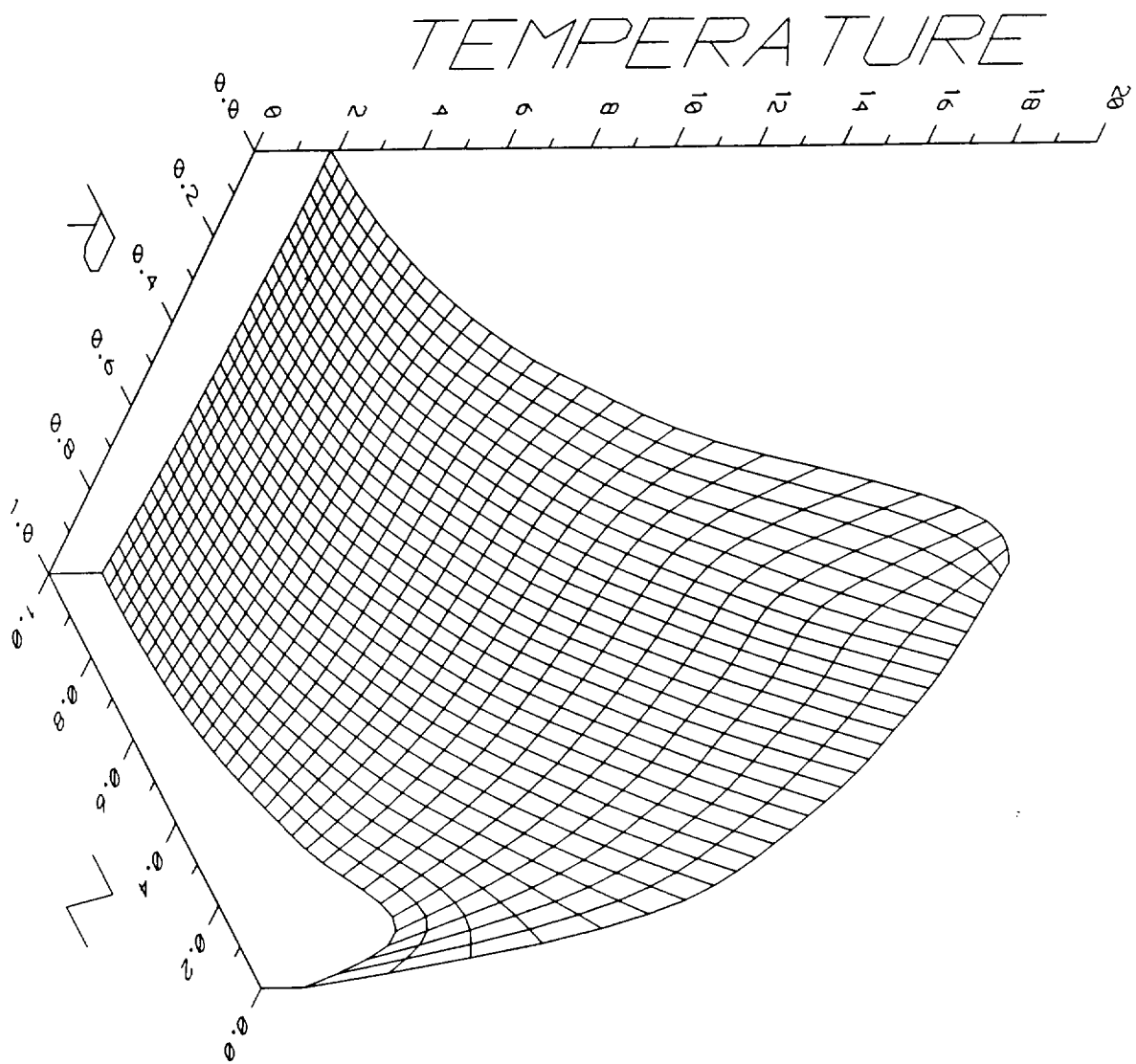


Figure A12. Normalized Solid Temperature at $\tau = 4.0$

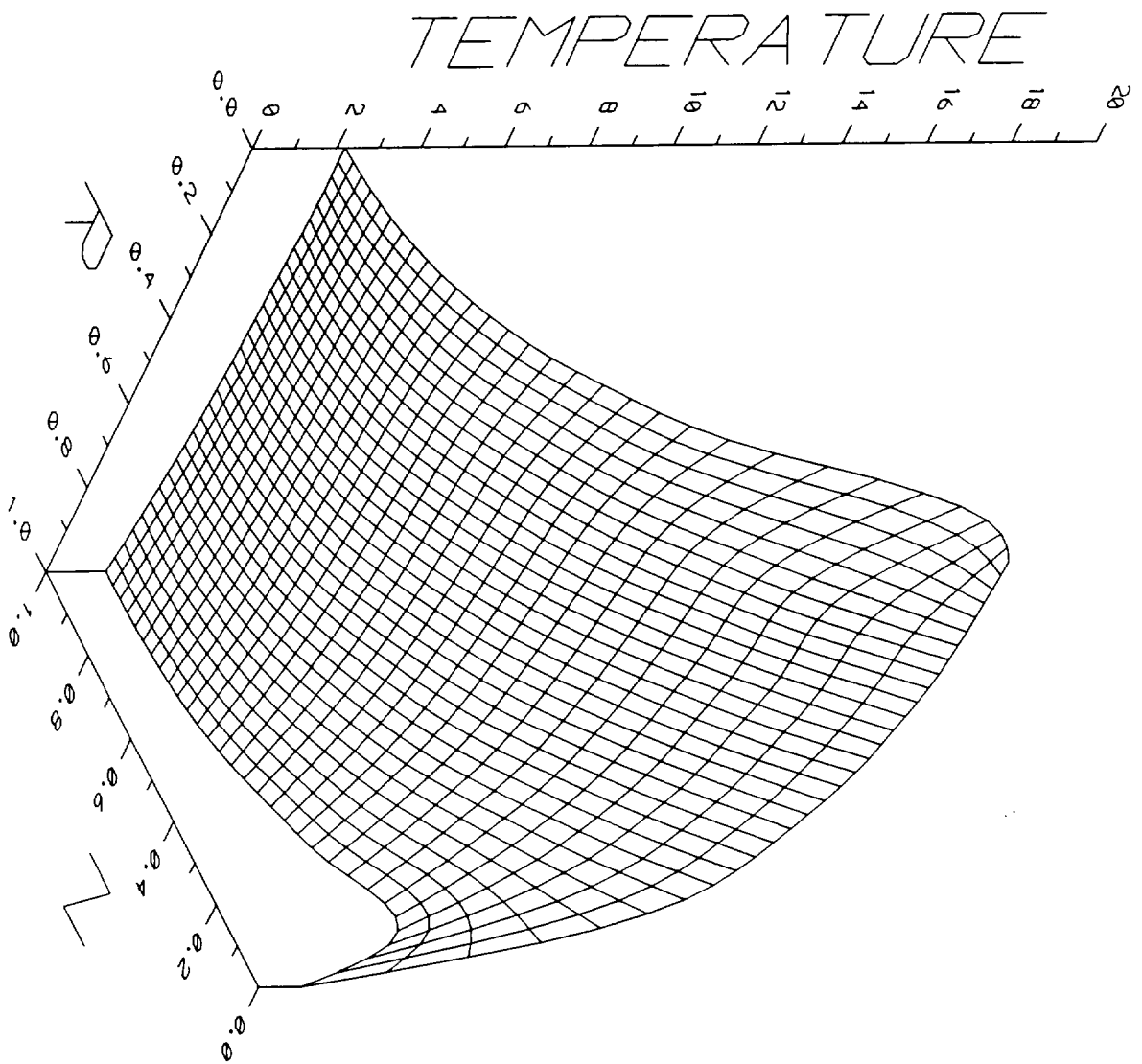


Figure A13. Normalized Solid Temperature at $\tau = 5.0$

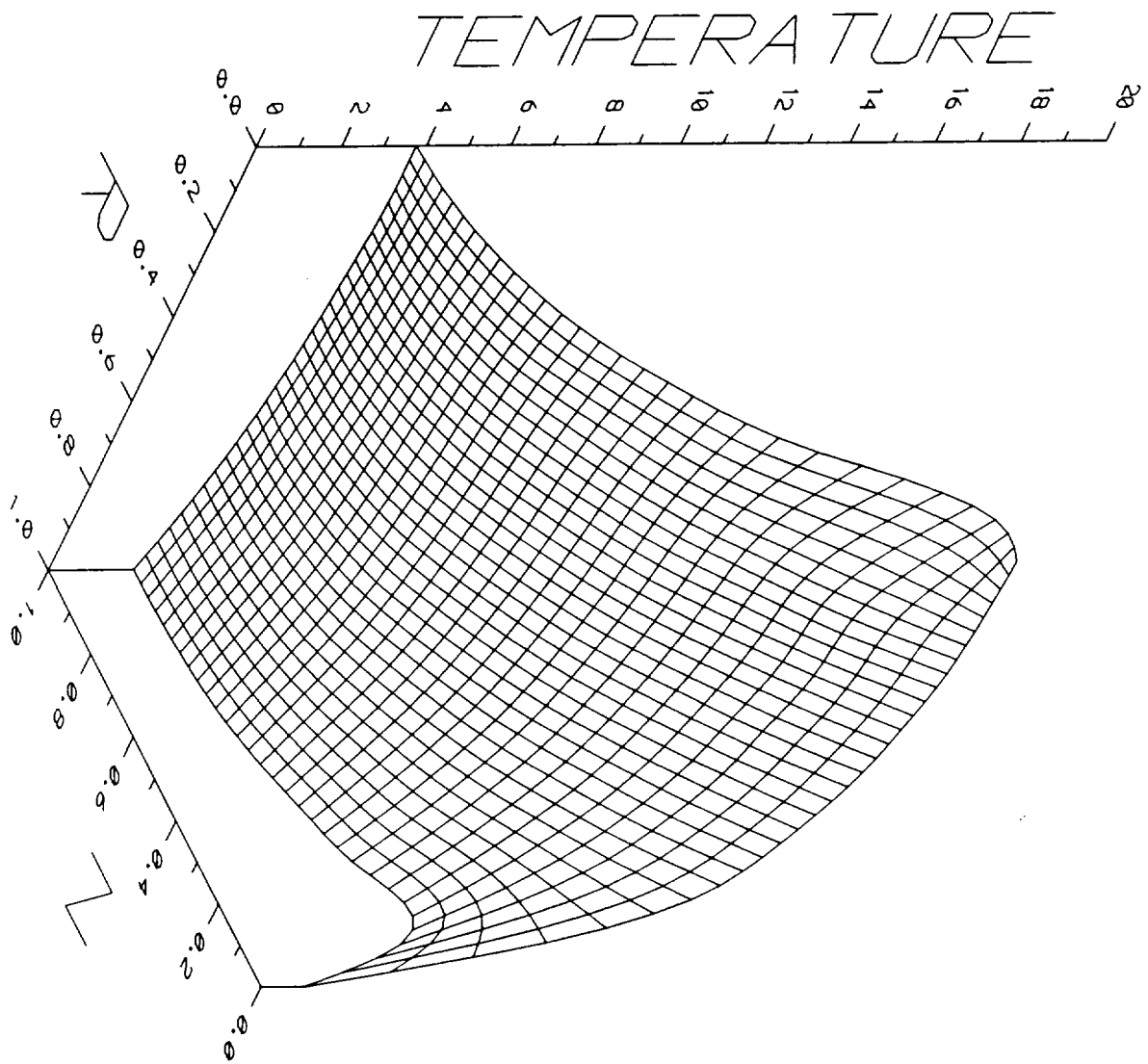


Figure A14. Normalized Solid Temperature at $\tau = 10.0$

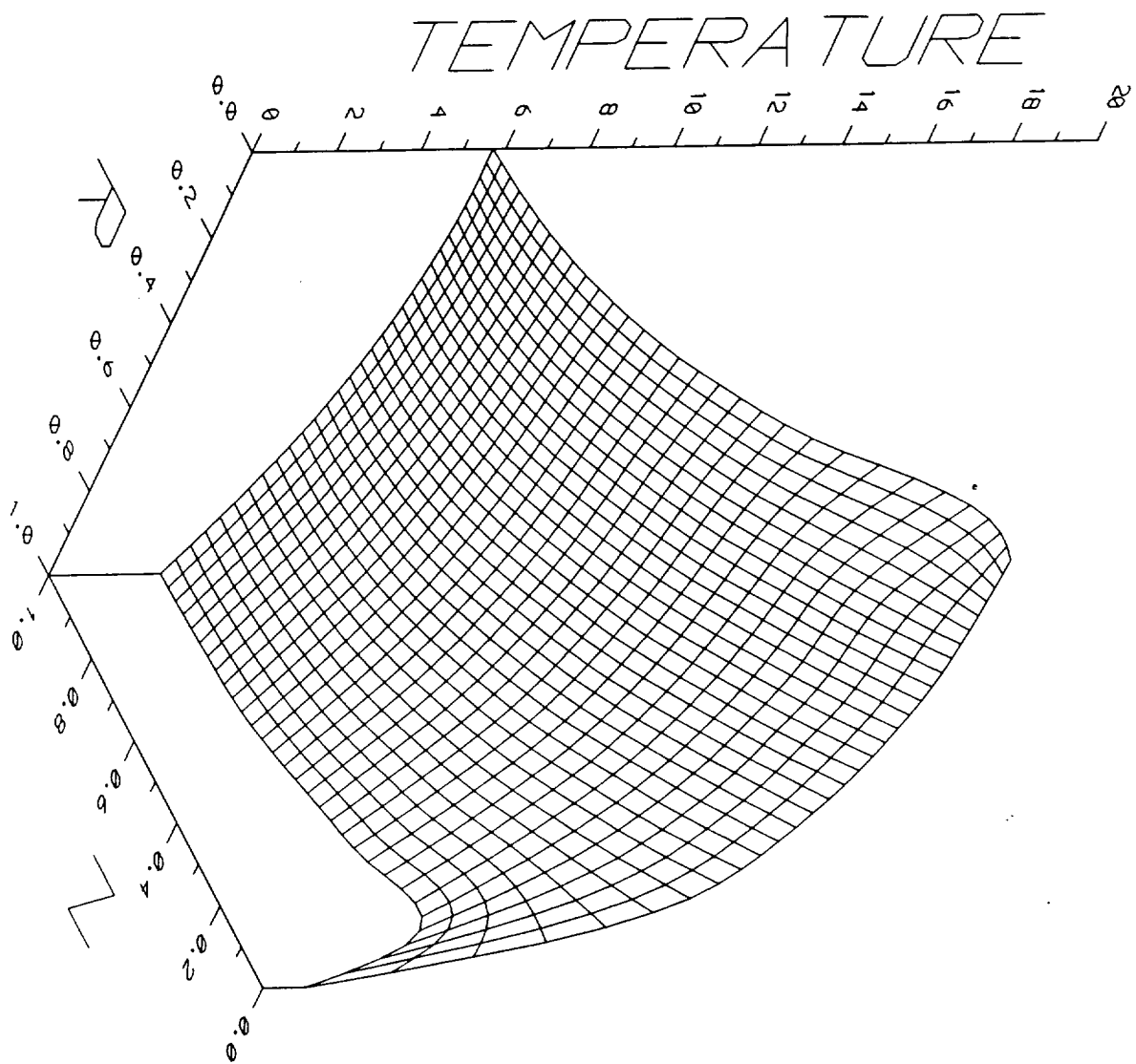


Figure A15. Normalized Solid Temperature at $\tau = 20.0$

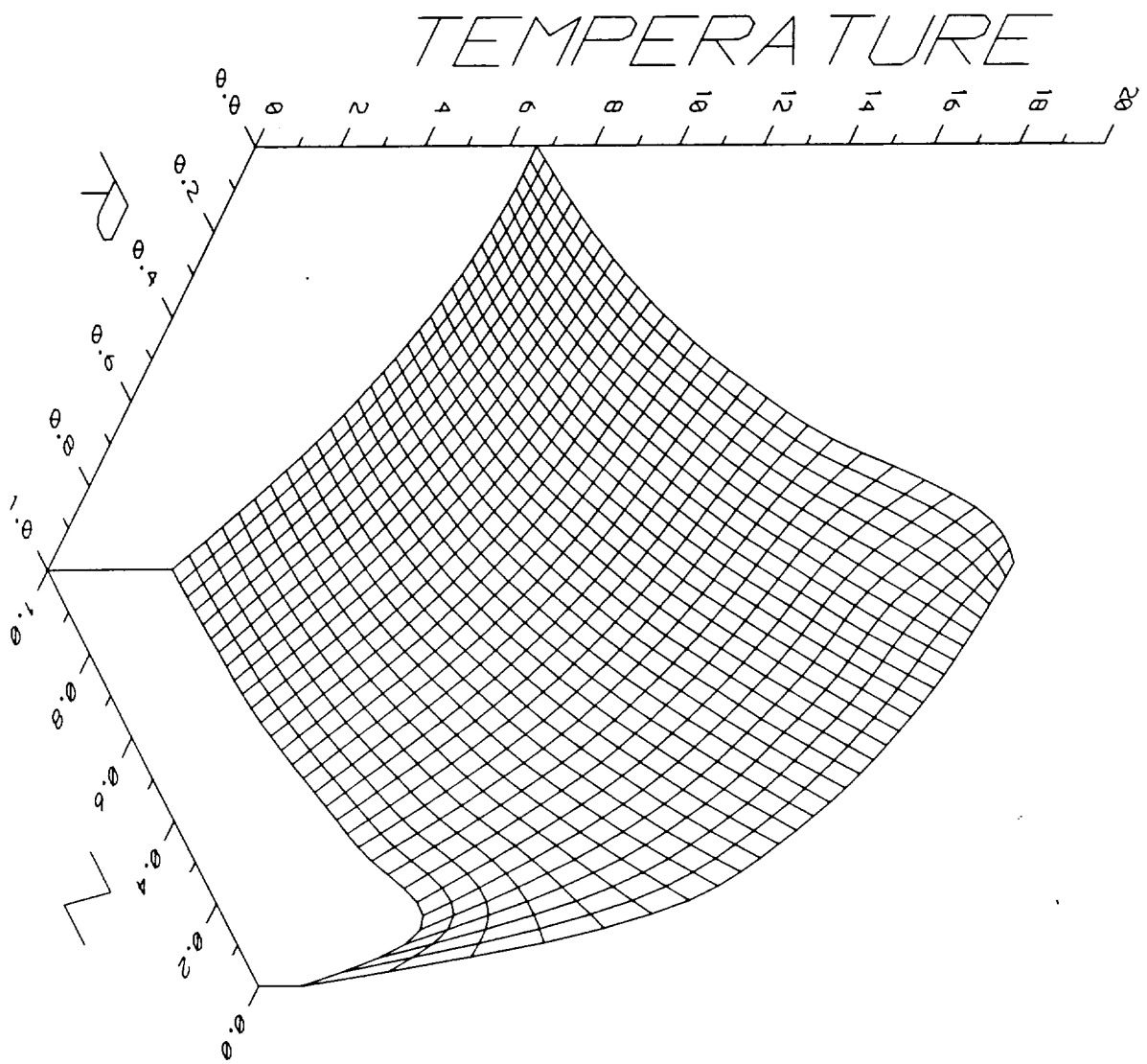


Figure A16. Normalized Solid Temperature at $\tau = 30.0$

